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TIDAL DISSIPATION IN THE OCEANS: ASTRONOMICAL, GEOPHYSICAL AND OCEANOGRAPHIC CONSEQUENCES

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The most precise way of estimating the dissipation of tidal energy in the oceans is by evaluating the rate at which work is done by the tidal forces and this quantity is completely described by the fundamental harmonic in the ocean tide expansion that has the same degree and order as the forcing function. The contribution of all other harmonics to the work integral must vanish. These harmonics have been estimated for the principal M_2 tide using several available numerical models and despite the often significant difference in the detail of the models, in the treatment of the boundary conditions and in the way dissipating forces are introduced, the results for the rate at which energy is dissipated are in good agreement. Equivalent phase lags, representing the global ocean–solid Earth response to the tidal forces and the rates of energy dissipation have been computed for other tidal frequencies, including the atmospheric tide, by using available tide models, age of tide observations and equilibrium theory. Orbits of close Earth satellites are periodically perturbed by the combined solid Earth and ocean tide and the delay of these perturbations compared with the tide potential defines the same terms as enter into the tidal dissipation problem. They provide, therefore, an independent estimate of dissipation. The results agree with the tide calculations and with the astronomical estimates. The satellite results are independent of dissipation in the Moon and a comparison of astronomical, satellite and tidal estimates of dissipation permits a separation of energy sinks in the solid Earth, the Moon and in the oceans. A precise separation is not yet possible since dissipation in the oceans dominates the other two sinks: dissipation occurs almost exclusively in the oceans and neither the solid Earth nor the Moon are important energy sinks. Lower limits to the Q of the solid Earth can be estimated by comparing the satellite results with the ocean calculations and by comparing the astronomical results with the latter. They result in $Q > 120$.

The lunar acceleration $\dot{n}_{☾}$, the Earth's tidal acceleration $\ddot{\theta}_T$ and the total rate of energy dissipation $\dot{E}$ estimated by the three methods give

	$\dot{n}_{☾}$		$\ddot{\theta}_T$	$\dot{E}$
	$10^{-23}\,s^{-2}$	$''cy^{-2}$†	$10^{-22}\,s^{-2}$	10^{19} erg s^{-1}
astronomical based estimate	-1.36	-28 ± 3	-7.2 ± 0.7	4.1 ± 0.4
satellite based estimate	-1.03	-24 ± 5	-6.4 ± 1.5	3.6 ± 0.8
numerical tide model	-1.49	-30 ± 3	-7.5 ± 0.8	4.5 ± 0.5

The mean value for $\ddot{\theta}_T$ corresponds to an increase in the length of day of 2.7 ms cy^{-1}. The non-tidal acceleration of the Earth is $(1.8 \pm 1.0)\ 10^{-22}\,s^{-2}$, resulting in a decrease in the length of day of 0.7 ± 0.4 ms cy^{-1} and is barely significant. This quantity remains the most unsatisfactory of the accelerations.

The nature of the dissipating mechanism remains unclear but whatever it is it must also control the phase of the second degree harmonic in the ocean expansion. It is this harmonic that permits the transfer of angular momentum from the Earth to the Moon but the energy dissipation occurs at frequencies at the other end of the tide's spatial spectrum. The efficacity of the break-up of the second degree term into the higher modes governs the amount of energy that is eventually dissipated. It appears that the break-up is controlled by global ocean characteristics such as the ocean–

† In this paper, cy is used as the symbol for century.

continent geometry and sea floor topography. Friction in a few shallow seas does not appear to be as important as previously thought: New estimates for dissipation in the Bering Sea being almost an order of magnitude smaller than earlier estimates. If bottom friction is important then it must be more uniformly distributed over the world's continental shelves. Likewise, if turbulence provides an important dissipation mechanism it must be fairly uniformly distributed along, for example, coastlines or along continental margins. Such a global distribution of the dissipation makes it improbable that there has been a change in the rate of dissipation during the last few millennium as there is no evidence of changes in ocean volume, or ocean geometry or sea level beyond a few metres. It also suggests that the time scale problem can be resolved if past ocean–continent geometries led to a less efficient breakdown of the second degree harmonic into higher degree harmonics.

1. Introduction

Tidal dissipation and its consequences on the lunar orbit and Earth rotation has become a classic problem yet there is probably no other subject in geophysics that has had as long a history of frustration and still attracts very considerable attention from geophysicists, astronomers and oceanographers. That this is so is as much a reflection of a fascinating subject as an indication of a problem of some importance in understanding the origin and dynamical evolution of the Moon. In his Harold Jeffreys lecture entitled 'Once again – tidal friction', Walter Munk introduced the subject by saying that in 1920 it appeared Jeffreys had solved the problem of tidal dissipation but that we have gone backwards ever since (Munk 1968). Now, some ten years later, we can say that we have gone full circle, for once again there is agreement between observations and theory of the secular acceleration of the Moon and the estimates of the dissipation of tidal energy in the oceans that cause this acceleration. Future new developments may have as a consequence that we have to go through the cycle of agreement and disagreement once again before we can finally conclude that the subject is closed. But if these results, such as those that may come from lunar laser ranging analysis, disagree with our present knowledge we can always use Jeffreys's dictum '(The analysis) covers only a short interval of time and will probably be improved' (Jeffreys 1973).

Recent improvements in the question of tidal dissipation and lunar orbit evolution include:

(i) Revised estimates of the recent (since the seventeenth century) astronomical data of observations of the Moon, Sun and Mercury, including improvements in the planetary ephemeris.

(ii) Re-evaluation of the ancient astronomical records in particular the solar eclipse observations, the enlargement of the reliable data set and the extension of this data set further back into time to about −1400.

(iii) Improvements in the ocean tide models by numerical modelling.

(iv) Recognition that the dissipation in the oceans is fully described by second degree harmonics in the ocean tide expansion, making the computation of dissipation by evaluating the rate at which work is done on the ocean surface, the most precise and direct method.

A number of new developments may result in further improvements in the near future:

(i) The widespread use of seafloor tide gauges in strategic positions on the seafloor will lead to further improvements in the ocean tide models.

(ii) Use of tidal parameters perturbing the orbits of close Earth satellites can be applied directly to the lunar problem.

(iii) The lunar laser range data will provide a check, or improve upon, the astronomical estimate of the present rate of the lunar acceleration.

The importance in studying the problem of tidal dissipation lies in its relevance to the study of the origin of the Moon and perhaps, by extrapolation, to the evolution of other planetary satellite systems. The problem of integrating the present lunar orbit into the past under the influence of tidal dissipation is well known and goes back to G. H. Darwin in the late nineteenth century. More recent studies are by Gerstenkorn (1955, 1969), Slichter (1963), MacDonald (1964) Kaula (1964) and Goldreich (1966). The consequences of this backward extrapolation are also well known; if the present rate of dissipation is representative of the past, the Moon will have been within the Roche limit of the Earth about 1.5 billion years ago. Neither the Earth's nor the Moon's surface shows evidence for such a geologically recent catastrophic event that a lunar sejour within the Roche limit is generally supposed to imply: the youngest rocks found on the Moon have been crystallized at least 3 billion years ago, while stromatalites indicate that lunar tides existed on the Earth 2.5 billion years ago and earlier. Recent discussions on the constraints imposed by tidal dissipation on theories on the lunar origin are given by Kaula (1971) and Kaula & Harris (1975).

The future evolution of the lunar orbit is perhaps of less immediate consequence for when the length of day has become equal to the lunar month and the Moon has begun its long spiralling motion back to the Earth, several billion years will have passed (Jeffreys 1929). But such an evolution is of interest for other planets whose satellites may already have passed through this stage as has been proposed by McCord (1968) and others in an attempt to explain the absence of large satellites around Venus and Mercury.

Clearly in any extrapolation, either into the past or into the future, the manner in which the tidal energy is dissipated is a crucial element in the theory. As Munk (1968) states 'Those who have been heavily involved in calculating past orbits have a vested interest in dissipation by bodily tides rather than ocean tides, for the solid Earth is less ephemeral than the ocean basins' but all the evidence points to the fact that the oceans are a much more important energy sink. The actual mechanics of the dissipation remains unclear nevertheless. If, as generally supposed since the work of Taylor and Jeffreys, the dissipation is by friction in shallow seas, then any extrapolation into the past becomes very uncertain indeed since we know that important changes in the ocean configuration have occurred in the past. Lambeck's (1975) recognition that the amount of dissipation is contained in only some of the second degree harmonics of the ocean tide points to a more uniform dissipating mechanism unless the shallow seas have an inordinate influence on the phase of the global ocean tide. As discussed below this is unlikely and ocean dissipation may not be as ephemeral as Munk suggests and the arguments indicate that the extrapolation into the past may still be valid. In particular, the modelling of the tide as a lagged ellipsoid remains a valid means of representing the ocean tide throughout the past history of the oceans. Only the lag angle may have changed but possibly not by great amounts. Thus even if the time scale of the orbital evolution cannot be established, extrapolation of the variations in inclination or eccentricity of the orbit as a function of the semi-major axis remains valid. This has not always been recognized. Munk (1968), for example, argues that MacDonald's (1964) results must be revised if shallow seas were important in the past. Goldreich (1966) also argues that possible intense local dissipation in the past may vitiate his results and Hipkin (1975) criticizes the use of the lagged tidal bulge as representing the ocean tide in studies of the tidal perturbations in satellite motions. Gold & Soter (1969), in the similar problem of the Venusian spin-orbit coupling, recognized that no matter what the form of the tide, in this case the atmospheric tide, only second degree terms contribute to the tidal torques. MacDonald (1964) and Gold (1975) have also stressed the importance of these terms in the Earth–Moon problem.

A second related study of some geophysical significance is the secular acceleration of the Earth's spin. The exchange of angular momentum between Earth and Moon results in a tidal deceleration of the Earth that is actually greater than that observed over the last 3000 years: Hence there is some mechanism that tends to accelerate the Earth. One proposal (Dicke 1966) is that a secular change in the gravitational constant is responsible for this part of the observed acceleration. A second proposal (Dicke 1966; O'Connell 1971) is that this acceleration is a consequence of the isostatic post glacial rebound, and a third proposal (Yukutake 1972) is that it is caused by electromagnetic core-mantle coupling. Other proposals invoke a growth of the core or variations in the depth of the principal mantle transition zones. Clearly a precise determination of the acceleration of the Earth's spin may provide important constraints on cosmological theories and geophysical models. Evidence for the non-tidal acceleration however, remains weak due to it being the relatively small difference between two larger quantities that are both known to within about 10% only. Muller (1976) concludes in fact that the present astronomical evidence does not support the existence of such an acceleration.

2. The problem

The effect of the Earth's tidal bulge on the lunar motion and on the Earth's tidal acceleration is well known and discussed generally in most geophysics textbooks (for example, Jeffreys 1962; Kaula 1968; Stacey 1969). A more detailed treatment of the problem is required here.

The potential U of the gravitational attraction at $\boldsymbol{r}$ due to a mass $m_☾$ at $\boldsymbol{r}_☾$ is

$$U = Gm_☾[1/|\boldsymbol{r}-\boldsymbol{r}_☾| - (\boldsymbol{r}_☾\cdot\boldsymbol{r}/r_☾^3)]$$

where G is the gravitational constant. If the geocentric angle between the positions $\boldsymbol{r}_☾$ and $\boldsymbol{r}$ is denoted by S, or $S = \arccos(\boldsymbol{r}_☾\cdot\boldsymbol{r}/r_☾ r)$, then apart from a constant the potential U can be expressed in terms of a series of Legendre polynomials as

$$U = \frac{Gm_☾}{r_☾}\sum_{l=2}^{\infty}\left(\frac{r}{r_☾}\right)^l P_{l0}(\cos S). \quad (1a)$$

Thus the tidal potential outside of the Earth $U_l(r)$ of the Moon's (or Sun's) gravitational attraction on the Earth is a harmonic function of degree l and can be written in the form

$$U_l(\boldsymbol{r}) = r^l W_l(\varphi,\lambda),$$

where $W_l(\varphi,\lambda)$ is a surface harmonic. At the Earth's surface $r = R$

$$U_l(\boldsymbol{R}) = (R/r)^l\, U_l(\boldsymbol{r}),$$

and the additional potential $\Delta U_l(R)$ due to the Earth's deformation is, by definition of the Love numbers k_l, $k_l U_l(R)$. Outside the Earth

$$\Delta U_l(\boldsymbol{r}) = k_l(R/r)^{l+1}\, U_l(\boldsymbol{R})$$
$$= k_l(R/r)^{2l+1}\, U_l(r). \quad (1b)$$

Equation (1b) describes the potential of the tidal deformation if the Earth's response is elastic. But the actual behaviour is anelastic and the response to the tidal potential $U_l(\boldsymbol{r})$ is delayed by an amount Δt. Thus the maximum deformation is reached at a time Δt after the Sun or Moon has passed through the observer's meridian. During this interval the Earth has rotated through an angle $\dot{\theta}\Delta t$ while the Moon has moved through a smaller angle $n_☾ \Delta t$. $\dot{\theta}$ is the siderial rotation of the Earth and $n_☾$ is the mean motion of the Moon. Viewed from space, the tidal bulge is ahead of the Moon by approximately an angle $\delta = (\dot{\theta} - n_☾)\,\Delta t$. This delay or lag can be most conveniently described by the difference in position between a fictitious position $\tilde{\boldsymbol{r}}_☾$ immediately overhead the bulge, and the actual position of the Moon $\boldsymbol{r}_☾$ at the same instant. The gravitational attraction of the Moon on this displaced bulge will exert a torque on the Earth whose value, integrated over the volume v_{E} of the Earth

$$\int_{v_{\mathrm{E}}} \rho \boldsymbol{x} \wedge \operatorname{grad} \boldsymbol{U}_l \, \mathrm{d}v_{\mathrm{E}}$$

and averaged over one cycle of the lunar motion, will not vanish. In consequence the Earth's spin decreases. The equal but opposite torque of the bulge on the Moon,

$$\int_{v_☾} \rho \boldsymbol{x} \wedge \operatorname{grad} \Delta \boldsymbol{U}_l \, \mathrm{d}v_☾$$

adds to the latter's orbital momentum an amount equal to that lost by the Earth and pushes the Moon away from the Earth in an ever increasing orbit until eventually other stabilizing factors come into play. The related quantities of interest that can be either observed or computed are (i) the rate of increase of the dimension of the lunar orbit or the secular acceleration of the Moon in longitude, (ii) the tidal acceleration of the Earth's spin, (iii) the amount of momentum and energy transferred from the Earth's spin to the lunar orbital motion and (iv) the amount of energy that must be dissipated in the Earth–Moon system.

Once the above torques are evaluated, the changes in the rotation of the Earth follow from Euler's equations. A more convenient method to estimate the accelerations is to evaluate directly the total angular momentum in the system; that associated with the orbital motion of the Earth and Moon about their centre of mass is given by

$$H_☾ = \frac{Mm_☾}{M+m_☾} r_☾^2 \dot{f}_☾ \approx \frac{Mm_☾}{M+m_☾} a_☾^2 (1-e_☾^2)^{\frac{1}{2}} n_☾,$$

where $f_☾$ is the true anomaly of the lunar motion. The component parallel to the rotation axis is $H_☾ \cos i_☾$. The angular momentum of the Earth's spin $\dot{\theta}$ is

$$H = C\dot{\theta},$$

and that of the Moon's spin is $C_☾ \dot{\theta}_☾$. The latter is negligibly small since

$$C_☾/C = m_☾ R_☾^2/MR^2 < 10^{-3}$$

and

$$\dot{\theta}_☾/\dot{\theta} \approx \tfrac{1}{27}.$$

If C, M and $m_☾$ do not vary with time and if there is no variation in the gravitational constant G, the conservation of angular momentum in the Earth–Moon system

$$H_☾ \cos i_☾ + C\dot{\theta} = \text{constant} \tag{2a}$$

requires that

$$\frac{1}{C}\frac{Mm_☾}{M+m_☾}\frac{\mathrm{d}}{\mathrm{d}t}[a_☾^2 n_☾ (1-e_☾^2)^{\frac{1}{2}} \cos i_☾] + \ddot{\theta}_{\mathrm{T}} = 0$$

or

$$\ddot{\theta}_{\mathrm{T}} = \frac{1}{C}\frac{Mm_☾}{M+m_☾} n_☾ a_☾^2 \left\{\tfrac{1}{3}\cos i_☾ \frac{\dot{n}_☾}{n_☾} + e_☾ \cos i_☾ \dot{e}_☾ + \sin i_☾ \frac{\mathrm{d}i_☾}{\mathrm{d}t}\right\}. \tag{2b}$$

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In these expressions $i_☾$ represents the inclination of the lunar orbit on the equator averaged over one period of the lunar ascending node on the ecliptic (Kaula 1964). These expressions also assume that there is no tidal effect on the ascending node but, as discussed below, a small tidal rate does exist. The $\ddot{\theta}_T$ represents the Earth's tidal acceleration due to the transfer of angular momentum from the Earth's spin to the lunar orbit. The $\dot{n}_☾$, $\dot{e}_☾$ and $di_☾/dt$ follow from the substitution of the tidal potential into the Lagrangian equations of motion (see below). Because of the Earth's hydrostatic response to changing forces, its mass distribution and hence its polar moment of inertia, C will be a function of the speed of rotation but this becomes important only when the history of the Earth's rotation is extrapolated back over geological time. The lunar torque $L_☾$ on the Earth follows from

$$L_☾ = C\ddot{\theta}_T.$$

The solar gravitational field also raises a tide on the Earth's surface and results in a further torque that does not vanish over one cycle of motion. In this case the transfer of angular momentum between the Earth and Sun is infinitesimally small, even when integrated over the age of the solar system, so that the Earth's orbit has undergone little tidal evolution. But the Sun's torque will accelerate the Earth's spin by a significant amount and must be included in discussing the spin history and energy dissipation questions. Thus a term $H_\odot \cos i_\odot$ must be added to equation (2a).

The rate at which tidal energy is dissipated in the Earth–Moon system can also be calculated in several equivalent ways. These include (i) to compute the time average of the rate of work done by the Moon on the Earth, (ii) to compute the rate of work done by the Earth on the Moon and (iii) to compute the energy balance between the spin and orbital motions. This last method, involving directly the quantities $\ddot{\theta}_T$ and $\dot{n}_☾$ does not require a knowledge of the energy sink and follows directly from the astronomical data.

The rotational energy associated with the Earth's spin is

$$E_1 = \tfrac{1}{2}C\dot{\theta}^2$$

and the rate of change of rotational energy is

$$dE_1 = d(\tfrac{1}{2}C\dot{\theta}^2)/dt = C\dot{\theta}\ddot{\theta}_T,$$

where $\ddot{\theta}_T$ is the total tidal acceleration of the Earth, that is $\ddot{\theta}_{T_☾} + \ddot{\theta}_{T_\odot}$. The energy associated with the orbital motion is

$$E_2 = \tfrac{1}{2}a_☾^2 n_☾^2 \frac{m_☾ M}{m_☾ + M} - \frac{GMm_☾}{a_☾} = -\frac{GMm_☾}{2a_☾}$$

and includes the potential and kinetic energies. The change in this energy state is

$$dE_2/dt = -\tfrac{1}{3}m_☾ n_☾ a_☾^2 \dot{n}_☾,$$

and represents about $n_☾/\dot{\theta}$ or 4% of the total spin energy. The energy transferred to the Earth orbit follows from the solar equivalent to dE_2/dt and is less than 1% of the spin energy transferred to the lunar orbit, or less than 5×10^{-4} of the spin energy lost. The total amount of energy dissipation is

$$\frac{dE}{dt} = \frac{dE_1}{dt} + \frac{dE_2}{dt} = C\dot{\theta}\ddot{\theta} - \tfrac{1}{3}m_☾ n_☾ a_☾^2 \dot{n}_☾ \quad (3a)$$

$$= (5.86\ddot{\theta} - 9.58\dot{n}_☾)\ 10^{40}\ \text{erg s}^{-1}. \quad (3b)$$

The astronomically observed quantity is $\dot{n}_☾$ from which the Earth's tidal acceleration can be partially estimated through the first term on the right hand side of equation (2b). Smaller

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contributions must be added to allow for (i) changes in the eccentricity and inclination of the lunar orbit and (ii) solar tides. These additional terms cannot be observed directly and are estimated from tidal energy dissipation calculations once the energy sink has been located (Lambeck 1975).

There are three approaches to estimating the above mentioned quantities which relate to the secular accelerations in the Earth's spin and in the lunar motion. The first is from the analysis of astronomical observations of the Moon's motion with respect to the Earth. These observations include the ancient eclipses as first recognized by Edmund Halley in 1695, occultation and meridian circle observations of the Moon and Sun from about 1600 onwards (Spencer-Jones 1939) and eventually lunar laser range observations (Williams 1977). The second method is to evaluate the tidal energy dissipation in the world's oceans as first attempted by Jeffreys (1920) and Heiskanen (1921). This method has generally been considered to be much less precise than the first approach, based upon the analysis of the astronomical data, but the most recent calculations by Lambeck (1975) indicate that this is no longer the case: The tidal dissipation calculations are as precise as the present astronomical data. The third method is to apply the results for tidal parameters estimated from close Earth satellite orbits directly to the lunar problem since the same parameters that cause short period perturbations in the satellite orbits also describe the secular evolution of the lunar orbit (Lambeck 1975; Lambeck & Cazenave 1977). This method has several advantages; (i) no assumptions need to be made as to where dissipation occurs in the Earth, (ii) it enables a separation of the amount of dissipation that occurs in the Earth from that occurring in the Moon, should the latter be significant and (iii) it enables the accelerations to be estimated separately for each tidal frequency of both lunar and solar tides. Results obtained from the satellite analysis, although only preliminary, are in essential agreement with those obtained by the other two methods.

3. The astronomical evidence

Astronomers observe the transits of stars across an observatory's meridian to establish a time scale referred to as universal time and compare this with a time scale that is supposedly uniform. Since 1955, this uniform time scale has been provided by atomic frequency standards and is referred to as atomic time. Before this, a dynamical time scale was used that is based upon the observed motions of the Sun, Moon and planets and satisfies the Newtonian equations of planetary motions. This scale, referred to as 'ephemeris time' is considerably less precise than the atomic time (Munk & MacDonald 1960) and perhaps also less uniform due to possible errors in the planetary theory. Evidence in the astronomical record for the tidal accelerations span a time interval of more than 3000 years and the discussion of the data is conveniently divided into three parts; (i) the data referenced to atomic time from 1955 to the present, (ii) the pre-atomic time telescope observations spanning an interval from about 1670 to 1955 and (iii) the ancient and medieval, or pre-telescope, observations going back to the thirteenth century B.C.

The observed telescopic quantities are the systematic discrepancies between the observed and computed longitudes of either the Moon, Sun or planets where the observed positions are referenced to the universal time scale kept by the irregularly rotating Earth and the computed positions are based on a gravitational theory that assumes a uniform time scale. These longitude discrepancies, apart from observational and computational errors, can arise in consequence of two phenomena; (i) irregular rotation of the Earth $\ddot{\theta}(T)$, including a secular part and long period (several decades and longer) fluctuations, resulting in an apparent acceleration in longitude of the observed bodies, and (ii) secular accelerations $\dot{n}$ of the observed body. If $\tau(T)$ is the amount by which the Earth is slow compared to the uniform time scale, or

$$\tau(T) = -\iint (\ddot{\theta}(T)/\dot{\theta})\, \mathrm{d}T\,\mathrm{d}T,$$

the Sun will have moved through an angle

$$\Delta\lambda_{\odot} = -n_{\odot}\iint (\ddot{\theta}(T)/\dot{\theta})\, \mathrm{d}T\,\mathrm{d}T \tag{4}$$

or

$$\Delta\lambda_{\odot} = a_{\odot} + b_{\odot}T_{☾} + \tfrac{1}{2}c_{\odot}T^2 + \beta(T), \tag{5}$$

where $\beta(T)$ represents the irregular non-secular part of $\ddot{\theta}(T)$. For observations of another planet, Mercury for example, whose mean motion is $n_{☿}$, we would expect an apparent longitude discrepancy of

$$\Delta\lambda_{☿} = (n_{☿}/n_{\odot})\,(a_{\odot} + b_{\odot}\,T + \tfrac{1}{2}c_{\odot}\,T^2) + (n_{☿}/n_{\odot})\,\beta(T), \tag{6}$$

if only $\ddot{\theta}(T)$ is responsible. Within the observational errors this is what is observed (Spencer-Jones 1939). Thus neither the Sun nor Mercury appears to undergo measurable non-gravitational accelerations in longitude. This is to be expected if these accelerations are of tidal origin for (equation (27))

$$\dot{n}_{\odot}/\dot{n}_{☾} = \frac{n_{\odot}}{n_{☾}}\frac{a_{☾}}{a_{\odot}}\frac{\dot{a}_{\odot}}{\dot{a}_{☾}} < 10^{-6}.$$

In consequence of the tidal interaction between the Earth and Moon, the latter experiences a real acceleration $\dot{n}_{☾}$ in longitude, resulting in a further longitude discrepancy of

$$\iint \dot{n}_{☾}(T)\,\mathrm{d}T\,\mathrm{d}T = a' + b'\,T + \tfrac{1}{2}c'\,T^2 \tag{7}$$

and added to that due to $\ddot{\theta}(T)$ gives a total discrepancy of

$$\Delta\lambda_{☾} = \iint [-n_{☾}\ddot{\theta}(T)/\dot{\theta} + \dot{n}_{☾}(T)]\,\mathrm{d}T\,\mathrm{d}T \tag{8}$$

$$= a_{☾} + b_{☾} + \tfrac{1}{2}c_{☾}\,T^2 + (n_{☾}/n_{\odot})\,\beta(T) \tag{9}$$

with

$$\begin{bmatrix} a_{☾} \\ b_{☾} \\ c_{☾} \end{bmatrix} = \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} + \frac{n_{☾}}{n_{\odot}}\begin{bmatrix} a_{\odot} \\ b_{\odot} \\ c_{\odot} \end{bmatrix}. \tag{10}$$

Observations of $\Delta\lambda_{☾}$ and $\Delta\lambda_{\odot}$ or $\Delta\lambda_{☿}$ at different epochs permit the constants in equations (5) and (9), as well as the function $\beta(T)$ to be estimated, in principle at least. Some difficulties may arise if the time span of the data analysed is comparable to the periods of the non-tidal rotational fluctuations, preventing an adequate separation of the quadratic term and $\beta(T)$ to be made. The usual approach (see, for example, Van der Waerden 1961) is to multiply (5) by $n_{☾}/n_{\odot}$ and to subtract it from (9). Then, with (10) and (7)

$$(\Delta\lambda_{☾} - (n_{☾}/n_{\odot})\,\Delta\lambda_{\odot}) = a' + b'\,T + \tfrac{1}{2}c'\,T^2$$

$$= \iint \dot{n}_{☾}(T)\,\mathrm{d}T\,\mathrm{d}T \tag{11}$$

or

$$\dot{n}_{☾}(T) = c'. \tag{12}$$

From (8) follows $$\ddot{\theta}(T) = (\dot{\theta}/n_{☾})\,[-\Delta\ddot{\lambda}_{☾}(T) + \dot{n}_{☾}(T)],$$

and with (9) and (12) $$\ddot{\theta}(T) = -(\dot{\theta}/n_{\odot})\,(c_{\odot} + \ddot{\beta}), \qquad (13)$$

which represents the fluctuations in the Earth's acceleration. From (12) the lunar acceleration follows directly from the constant in the quadratic term in (11) and its determination should present little difficulty since $\beta(T)$ has been eliminated. The secular changes in rotation follow from the quadratic term in (5).

The telescope observations made since the seventeenth century, consist mainly of meridian transits of the Moon or occultations of stars by the Moon and of transits of Mercury across the Sun's disk. Their analysis essentially follows the above method but with only some three hundred years of data, it is problematical whether a true measure of the Earth's secular acceleration can be obtained. Observations of lunar occultations made since 1955, referenced to the atomic time scale, provide a direct measure of the real acceleration of the Moon in longitude as described by the quadratic expression (7) and (12).

Ancient chronicles provide valuable astronomical records of solar and lunar eclipses, of lunar and planetary occultations and conjunctions and of the times of equinox passage of the Sun. By using gravitational theories based on a uniform time scale, the times and positions at which these phenomena should have been observed can be predicted, and any differences with the observations provide a measure of the non-Newtonian accelerations (Munk & MacDonald 1960; Jeffreys 1962). Of the various records the most valuable ones are the solar eclipse magnitudes which give a relation between $\dot{n}_{☾}$ and $\ddot{\theta}$ of the type

$$\Delta\ddot{\lambda} = \frac{\dot{\theta}}{n_{☾} - n_{\odot}}\,\dot{n}_{☾} - \ddot{\theta},$$

where $\Delta\lambda$ is the difference between the observed and predicted longitudes of the eclipse.

The astronomical evidence for the accelerations in the Moon's longitude and in the Earth's rotation was reviewed by Munk & MacDonald in 1960 and apart from the new length-of-day data observed with respect to atomic time since 1955, any recent improvements in our knowledge of the accelerations has come from a re-analysis of essentially the same data, both the telescope data and the ancient chronicles. However, the various redeterminations have perhaps not greatly clarified our knowledge of the accelerations; various estimates for $\dot{n}_{☾}$ differ by a factor of three and more. With this state of confusion it is tempting to abandon the subject altogether until new data, from the lunar laser ranging program for example, become available, were it not for the fact that there does appear to be a convergence of opinion on how the available evidence should be interpreted and for the fact that laser ranging to the Moon will not give us the geophysically interesting non-tidal accelerations of the Earth. Furthermore, because of the long time interval covered by the eclipse records, any remaining periodic errors in the lunar theory will tend to have a minimum effect on the estimates of the accelerations which would not be the case for the very precise laser range observations collected over a much shorter time interval.

The lunar acceleration

The nominal value for the lunar acceleration $\dot{n}_{☾} = -22.4''\,\mathrm{cy}^{-2}$ is that attributed to Spencer-Jones (1939) (see also Clemence 1948) and is based on telescope observations made since 1680 of occultations of stars by the Moon, of longitudes of the Sun and transits of Mercury across the Sun. Spencer-Jones's results have been verified by K. P. Williams in 1940 and by Clemence in

1943, although Morrison (1972) argues that the uncertainty in $\dot{n}_{☾}$ may be about three times larger than estimated by Spencer-Jones. Also according to Morrison (1973), Spencer-Jones's value is strongly dependent on the early seventeenth century Mercury transits and Van Flandern (1975) uses this comment to discard this determination 'because of the high probability of systematic errors'. Morrison & Ward (1975) have reanalysed the Mercury observations for the years 1677–1973 giving a series of equations similar to (6) but also including unknowns referring

Table 1. Astronomical estimates of the lunar acceleration in longitude, $\dot{n}$ and of the Earth's secular acceleration, $\ddot{\theta}$

authors		$\dot{n}$(″cy^{-2})	$\ddot{\theta}$(″cy^{-2})[9]	corrected θ(″cy^{-2})
Spencer-Jones (1939)[1]		-22.4 ± 7[2]	—	—
Munk & MacDonald (1960)		—	-986[10]	—
Morrison & Ward (1975)		-26.0 ± 2[3]	—	—
		-28.0 ± 2[4]	—	—
Van Flandern (1970)		-52.0 ± 16	—	—
Van Flandern (1975)		-65.0 ± 18[5]	—	—
Morrison (1973)		-42.0 ± 6[13]	-830[13]	—
Oesterwinter & Cohen (1972)		-38.0 ± 4	—	—
Fotheringham[1]		-30.8	-1340	—
De Sitter[1]		-37.7	-1670	—
Dicke (1966)		(-22.4)[6]	-860	—
Curott (1966)		(-22.4)[6]	-770	-1400[11]
Newton (1970)		-42.0 ± 4[7]	-1200	-1350[11]
Newton (1972)		-79.0 ± 16	-2050	—
Stephenson (1972)		-34.0 ± 2	—	—
Muller & Stephenson	(1975)	-37.5 ± 5.0	-1390	-1390
Muller (1975)[8]	(i)	-34.5 ± 3.0	-1120 ± 60	—
	(ii)	-30.4 ± 3.0	—	—
	(iii)	-28.0 ± 2.0	—	—
Muller (1976)[12]	(i)	-30.0 ± 3.0	-1070 ± 50	—
	(ii)	-27.2 ± 1.7	—	—

(1) Quoted in Munk & MacDonald (1960).
(2) Standard deviation estimated by Morrison (1972).
(3) Mercury transits from 1677–1973.
(4) Mercury transits from 1789–1973.
(5) According to Muller (1975) this value should be reduced to $-36''$.
(6) Adopted value of Spencer-Jones.
(7) Mean of values determined for epochs -200 and $+1000$.
(8) See text for explanation of these three values.
(9) Original values found by the various authors.
(10) Based on telescope observations analysed by Spencer-Jones (1939) and Brouwer (1952).
(11) As corrected by Muller & Stephenson (1975).
(12) See text for the explanation of the two values for n.
(13) These values are highly correlated.

to corrections to some of the orbital elements of Mercury and the Sun. Combined with observations of the Moon, mainly occultations collected from several recent discussions, they find $\dot{n}_{☾} = -26''\,\mathrm{cy}^{-2}$, in essential agreement with the Spencer-Jones determination, the difference being due to the fact that Morrison & Ward have used additional data and that they have modified the orbital elements. To investigate the influence of the earlier observations, they have also derived a solution using the transit observations since 1789 only, yielding $-28''\,\mathrm{cy}^{-2}$, in agreement with their value based on the longer time span: There is no good reason for discarding the earlier observations.

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Van Flandern (1970) analysed occultations of stars by the Moon, the observed time of occultation being expressed with respect to atomic time. From 14 years of data, Van Flandern finds $-52'' \text{cy}^{-2}$ and, five years later, from nearly twenty years of data he finds $-65'' \text{cy}^{-2}$ (Van Flandern 1975). However, a comment by Muller (1975, page 12.3) indicates that Jeffreys is probably correct in his dictum, quoted in the first paragraph of this paper, for apparently upon correcting some of his data, Van Flandern has reduced his estimate to $-36'' \text{cy}^{-2}$. Morrison (1973) using essentially the same method with data from 1955 to 1972, finds $-42'' \text{cy}^{-2}$. Oesterwinter & Cohen (1972) have analysed meridian circle observations of the Sun, Moon and planets since 1913 in a general solution for orbital constants, the tidal accelerations and the Earth's irregular rotation, and find $\dot{n}_{☾} = -38'' \text{cy}^{-2}$.

These recent values based on data covering short time intervals tend to be consistently larger than the values based on nearly three hundred years of data and if real, this discrepancy is suggestive of a long period error in the orbital theories used, rather than any real change in the lunar acceleration, for reasons discussed below. Van Flandern (1975) attributes any difference between estimates for $\dot{n}_{☾}$ based on ephemeris time and on atomic time, to changes in the gravitational constant G, but in view of the uncertainty of his estimate such a conclusion is premature.

Various analyses of the ancient eclipse records show a similar spread of values for $\dot{n}_{☾}$. Fotheringham's classical solution as given by Munk & MacDonald (1960) is determined essentially from three eclipses, Plutarch, the Eponym canon and Hipparchus, although he did consider other records as well. As already indicated by Munk & MacDonald, the interpretation of the first two of the records is quite uncertain and for the last, three alternate dates are possible (see Muller 1975 for a recent discussion). De Sitter's analysis, as quoted by Munk & MacDonald covers similar data but with the addition of a Babylonian lunar eclipse. Dicke (1966) and Curott (1966) adopt Spencer-Jones' value for $\dot{n}_{☾}$ and use the eclipse observations to determine the non-tidal acceleration of the Earth. Dicke's solution rests heavily upon the eclipses of the Greek and Roman classics while the importance of Curott's study is his recognition of the valuable source of astronomical data in the Chinese historical records, covering a time span of at least 2000 years. These Chinese records have been extensively examined by Stephenson (1972).

Newton (1970) found two values for $\dot{n}_{☾}$, one centred at epoch -200 and the other centred at epoch $+1000$, from a detailed discussion of ancient and medieval records. These values were considerably larger than the 'modern' value of Spencer Jones and that which he found from analysing artificial satellite orbits (Newton 1968) and he suggested that the lunar acceleration may have changed considerably over the last 2500 years. In a second exhaustive study of the medieval records (Newton 1972), he finds a value that is considerably larger than his previous values and argues that an important change in the lunar acceleration occurred near the year $+700$. Physically, in view of the dissipation mechanism, this is improbable (see §8) and Muller & Stephenson (1975) argue that Newton's results are strongly biased by his use of partial eclipses (see also Muller 1975). Muller & Stephenson (1975) have re-analysed all available eclipse data going back to the thirteenth century B.C. and from a careful examination of the reliability of the records, they keep only those observations that are of total solar eclipses or that specifically deny totality. As a result they use a much smaller data set than does Newton but they argue convincingly that in this way they avoid several biases that are present in Newton's results. Muller (1975) uses the same material as in his study with Stephenson and any differences result from modifications in the astronomical estimation process. In all these recent studies there has been an important drift away from the Greek and Roman classical sources to more reliable historical records.

Muller's various values are summarized in table 1. His first value is obtained by assuming that both the theory and the initial conditions used in the lunar theory are correct. In the second solution he argues that if there is an error in the observed rate of the lunar node, used in the ephemeris, this will have important consequences on the solution for $\dot{n}_☾$. By using Martin & Van Flandern's (1970) correction $\dot{\Omega}_☾ = (4.3 \pm 0.4)''$ cy^{-2}, found from their analysis of lunar occultation data between 1780 and the present, the solution for $|\dot{n}_☾|$ is reduced by about 5″ cy^{-2}. Muller states that this remains the single most important source of possible systematic error in his solution. The third value is based on a number of different solutions which use, in addition to his eclipse data, the following information: (i) Morrison & Ward's (1975) result for $\dot{n}_☾$; (ii) Van Flandern's revised result of his 1975 analysis for $\dot{n}_☾$; (iii) Martin & Van Flandern's (1970) value for $\dot{\Omega}_☾$; (iv) a relation expressing the secular changes in rotation as the sum of the total tidal acceleration including the solar ocean and atmospheric tide contributions estimated by Lambeck (1975), variations in the gravitational constant G and the non-tidal acceleration of the Earth; (v) three alternative cosmologies. The solutions for these various subsets of data all give a lunar acceleration of about $-28''$ cy^{-2}.

More recently Muller (1976) has revised his solution for $\dot{n}_☾$, the non-tidal $\ddot{\theta}$ and the change in gravitational constant, using a correction of $\dot{\Omega} = 4.39 + 0.15''$ cy^{-1} deduced from an integration of the solar system 250 years back into time, by himself, Newhall, Van Flandern and Williams. Muller's revised estimate based on solar eclipses only is $-30.0''$ cy^{-1} and based on the additional data and a variable gravitational constant, is $-27.2''$ cy^{-1}.

Secular changes in the length of day

The telescopic data covers too short a time interval to give meaningful estimates of the secular acceleration of the Earth although Morrison's (1973) study covering the longest interval, from 1663 to 1972, determines a value for the change in the length of day of about 1.5 ms cy^{-1}. Dicke's (1966) and Curott's (1966) values are found by adopting the Spencer-Jones value for $\dot{n}_☾$. Newton's (1970) average value for the epochs -200 and $+1000$ is adopted in table 1. Muller & Stevenson's (1975) and Muller's (1975) values are also given. As stressed in these last two papers, the Earth's rotational acceleration, if constant, gives a parabolic function $\tau(T)$ according to equation (5) divided through by $n_\odot$. Then $\ddot{\theta}$ follows from $\dot{\theta}c_\odot/n_\odot$ (equation 13). Newton (1970, 1972), Curott (1966) and others have analysed $\tau(T)$ or equivalently $\lambda_\odot(T)$ by assuming that the linear part of the relation (5) vanishes at epoch 1900 whereas Muller & Stephenson show that it actually vanishes at about 1770. This oversight introduces significant errors in $\ddot{\theta}$, and the last column of table 1 gives the values as corrected by Muller & Stephenson. Muller's best estimate for $\ddot{\theta}$ is $-1120''$ cy^{-2} giving an average change in the length of day of 2.0 ms cy^{-1} over the last 3000 years. Muller's (1976) revised estimate is $-1070''$ cy^{-2}.

Summary

We adopt the following values for the accelerations

$$\dot{n}_☾ = -(28'' \pm 3)''\,\mathrm{cy}^{-2} = -(1.35 \pm 0.15)\ 10^{-23}\,\mathrm{rad\,s}^{-2},$$

$$\ddot{\theta} = -(1120 \pm 120)''\,\mathrm{cy}^{-2} = (-5.40 \pm 0.5)\ 10^{-22}\,\mathrm{rad\,s}^{-2}.$$

The lunar acceleration is essentially the mean of the recent results obtained from the telescope observations and from the eclipse records. As discussed in § 8, it is improbable that $\dot{n}_☾$ has varied significantly over the last three or four millennia so that these two data sources should provide comparable estimates. Better estimates of $\dot{n}_☾$ may result from (i) an improved lunar ephemeris, based in part on improved constants of integration and perhaps on an improved theory for some of the long period perturbations, (ii) additional eclipse records going further back into time than the currently available data and (iii) precise laser range observations. The above value for the Earth's secular acceleration is that obtained by Muller (1975).

For completeness we also give:

(i) the tidal acceleration of the Earth, $\ddot{\theta}_T$ which follows from equation (2*b*),

$$\ddot{\theta}_T = -7.22 \times 10^{-22}\,\mathrm{s}^{-2},$$

(ii) the non-tidal acceleration of the Earth, $\ddot{\theta}_{NT} = 1.8 \times 10^{-22}\,\mathrm{s}^{-2}$,

(iii) the rate of tidal energy dissipation; $\mathrm{d}E/\mathrm{d}t = 4.1 \times 19^{19}\,\mathrm{erg\,s^{-1}}$.

These three estimates are based on the astronomical observations, with corrections added to allow for secular changes in inclination and eccentricity, and for the solar tides (§8).

4. Tidal potentials

The solid tide

Kaula's (1964) development of the tidal potential as a function of orbital elements is used. The potential at $\boldsymbol{r}$ due to the tide raised by the Moon follows from (1*b*) with (1*a*), that is,

$$\Delta U(r) = \frac{Gm_☾}{r_☾} \sum_{l=2}^{\infty} k_l \left(\frac{R}{r_☾}\right) \left(\frac{R}{r}\right)^{l+1} P_{l0}(\cos S), \tag{14a}$$

where the geocentric angle S can be expressed in terms of spherical coordinates r, φ, λ of $\boldsymbol{r}$ and $r_☾$, $\varphi_☾$, $\lambda_☾$ of the Moon at $\boldsymbol{r}_☾$ by

$$\cos S = \sin\varphi \sin\varphi_☾ + \cos\varphi \cos\varphi_☾ \cos(\lambda - \lambda_☾). \tag{14b}$$

Then (14*a*) becomes (see also Kaula 1968)

$$\Delta U(r) = \frac{Gm_☾}{r_☾} \sum_{l=2}^{\infty} k_l \left(\frac{R}{r_☾}\right)^l \left(\frac{R}{r}\right)^{l+1} \sum_{m=0}^{l} (2-\delta_{0m}) \frac{(l-m)!}{(l+m)!} P_{lm}(\sin\varphi)\, P_{lm}(\sin\varphi_☾) \cos m(\lambda - \lambda_☾), \tag{15}$$

where $\delta_{0m} = 1$ for $m = 0$, otherwise $\delta_{0m} = 0$. The lunar position $r_☾$, $\varphi_☾$, $\lambda_☾$ at an instant T can be expressed as a function of orbital elements. If instantaneous Keplerian elements κ_i ($i = 1, 2, \ldots, 6$), referred to an equatorial reference frame, are used, where

$\kappa_1 = a_☾$ = semi-major axis,
$\kappa_2 = e_☾$ = eccentricity,
$\kappa_3 = i_☾$ = inclination of the lunar orbit on the equatorial plane,
$\kappa_4 = M'_☾$ = mean anomaly and $\dot{M}'_☾ = n_☾$ where $n_☾$ is the mean motion,
$\kappa_5 = \omega_☾$ = argument of perigee,
$\kappa_6 = \Omega_☾$ = longitude of the ascending node,

then the appropriate transformation from spherical to orbital elements follows as (see Kaula 1966)

$$\frac{1}{r}\left(\frac{R}{r}\right)^l P_{lm}(\sin\phi)\, e^{jm\lambda} = \frac{1}{a}\left(\frac{R}{a}\right)^l \sum_{p=0}^{l} F_{lmp}(i) \sum_{q=-\infty}^{\infty} G_{lpq}(e) \begin{bmatrix} e^{jv^☾_{lmpq}} \\ -je^{jv^☾_{lmpq}} \end{bmatrix}_{l-m\,\mathrm{odd}}^{l-m\,\mathrm{even}} \tag{16}$$

with $v^☾_{lmpq} = (l-2p)\,\omega_☾ + (l-2p+q)\,M'_☾ + m(\Omega_☾ - \theta)$. The $F_{lmp}(i)$ and $G_{lpq}(e)$ are polynomials in $\sin i$ and e respectively. The latter are proportional to $e^{|q|}$ so that the summation over q need only be carried out over a small number of terms; that is, $q = 0, \pm 1, \pm 2$. As the factor $(R/r)^l \approx (\frac{1}{60})^l$ for the Moon and very much smaller for the Sun, only terms with $l = 2$ will be important at present although terms with $l = 3$ or 4 may have been important during a period of close approach. The potential $\Delta U(r)$ becomes

$$\Delta U(r) = \frac{Gm_☾}{a_☾} \sum_l \sum_m \left(\frac{R}{r}\right)^{l+1} \left(\frac{R}{a_☾}\right)^l k_l (2-\delta_{0m}) \frac{(l-m)!}{(l+m)!} \times P_{lm}(\sin\varphi) \sum_p F_{lmp}(i_☾) \sum_q G_{lpq}(e) \begin{bmatrix}\cos\\ \sin\end{bmatrix}_{l-m\ \text{odd}}^{l-m\ \text{even}} (v^☾_{lmpq} - m\lambda). \tag{17}$$

For the anelastic response we are concerned with the potential at r, φ, λ raised by the Moon at the fictitious position $\tilde{r}_☾, \tilde{\varphi}_☾, \tilde{\lambda}_☾$. This fictitious position can be transformed into fictitious Keplerian elements $\tilde{\kappa}$ defined by $\kappa + \dot{\kappa}\Delta t$, but in the short time interval Δt that it takes for the Earth to respond to the tidal attraction, the only element of the lunar orbit that will have changed by a significant amount, is the mean anomaly $M'_☾$ by $n_☾ \Delta t$ and the Earth will have rotated through an angle $\dot{\theta}\Delta t$. To a lesser extent, $\omega_☾$ and $\Omega_☾$ will have changed by small amounts due to the secular rates of these elements; $a_☾$, $e_☾$ and $i_☾$ will not have undergone any perceptible change. Thus to introduce the dissipation into equation (17) we need only substitute $\tilde{v}^☾$ for $v^☾$, where

$$\tilde{v}^☾_{lmpq} = (l-2p)\,\omega_☾ + (l-2p+q)\,M'_☾ + m(\Omega_☾ - \theta) + \epsilon_{lmpq}$$

with $$\epsilon_{lmpq} = [(l-2p)\,\dot{\omega}_☾ + (l-2p+q)\,n_☾ + m(\dot{\Omega}_☾ - \dot{\theta})]\,\Delta t \approx [(l-2p+q)\,n_☾ - m\dot{\theta}]\,\Delta t. \tag{18}$$

If the position at which the potential is evaluated refers to a satellite at r, φ, λ, it can also be expressed in terms of Kepler elements $a, e, i, \omega, \Omega, M'$ with a transformation similar to (16). The final form of the potential is

$$\Delta U(r) = \sum_{l=2}^{\infty} \sum_{m=0}^{l} \sum_{p=0}^{l} \sum_{q=-\infty}^{\infty} \sum_{j=0}^{l} \sum_{g=-\infty}^{\infty} \Delta U_{lmpqjg} \tag{19a}$$

with $$\Delta U_{lmpqjg} = k_l \left(\frac{R}{a_☾}\right)^l \left(\frac{R}{a}\right)^{l+1} \frac{Gm_☾}{a_☾} (2-\delta_{0m}) \frac{(l-m)!}{(l+m)!} F_{lmp}(i_☾) \times F_{lmj}(i)\, G_{lpq}(e_☾)\, G_{ljg}(e) \cos(v^☾_{lmpq} - v_{lmjg} + \epsilon_{lmpq}) \tag{19b}$$

and is equivalent in all respects to the form developed by Kaula (1964, 1969) and used by Goldreich (1966), Goldreich & Peale (1968), Peale (1973), Lambeck (1975) and others.

The tidal perturbations of the orbit of the satellite are found by substituting $\Delta U(r)$ into the Lagrange planetary equations of motion (Kaula 1966). For satellite studies, of greatest interest are the perturbations in inclination and in right ascension since these elements are generally most precisely determined, although all elements κ_i are perturbed to varying degree. The result for i due to ΔU_{lmpqjg} is for example

$$\left.\frac{di}{dt}\right|_{lmpqjg} = \frac{Gm_☾[(l-2p)\cos i - m]}{na^2(1-e^2)^{\frac{1}{2}}\sin i} k_l \frac{1}{a_☾} \left(\frac{R}{a_☾}\right)^l \left(\frac{R}{a}\right)^{l+1} (2-\delta_{0m}) \frac{(l-m)!}{(l+m)!} \times F_{lmj}(i)\, F_{lmp}(i_☾)\, G_{ljg}(e)\, G_{lpq}(e_☾) \sin(v^☾_{lmpq} - v_{lmjg} + \epsilon_{lmpq}). \tag{20}$$

These and the corresponding perturbations in the other elements have periods longer than one day only when $l - 2j + g = 0$, and for all other terms not satisfying this condition, the perturbations will be of small amplitudes since their frequencies are high. The frequencies of the long period tidal terms are governed by both the lunar and satellite motions around the Earth and the spectrum will differ quite considerably from that of the tidal variations observed at the Earth's surface in, for example, gravity. Also, as the amplitude of the orbital perturbations depends upon frequency, tidal terms, observed at the Earth's surface to be of small amplitude, may become important in the satellite spectrum if the two basic motions become commensurable. Thus by a careful selection of elements for an orbit, different fundamental tidal frequencies can be made to have more or less important effects on the satellite.

In the study of the consequences of the delayed tidal response on the lunar orbit itself we are concerned with the action of the potential (19) raised by the Moon at the fictitious position $\tilde{\boldsymbol{r}}_☾$ on the actual Moon at $r_☾$. This effect is found by substituting the potential into the Lagrangian equations and then equating the elements of the satellite position with those of the Moon. Furthermore, we are interested only in the secular perturbations, those with zero frequency, and for which $\dot{v}^☾_{lmpq}-\dot{v}^☾_{lmjg}=0$, or $p=j$ and $q=g$. Then for the three elements $a_☾$, $e_☾$, $i_☾$, which relate to the conservative quantities in the Earth–Moon system, dropping the subscripts ☾,

$$\dot{a}_{lmpq} = 2K_{lm}[F_{lmp}(i)]^2[G_{lpq}(e)]^2(l-2p+q)\sin\epsilon_{lmpq},$$

$$\dot{e}_{lmpq} = K_{lm}\frac{(1-e^2)^{\frac{1}{2}}}{ae}[F_{lmp}(i)]^2[G_{lpq}(e)]^2[(1-e^2)^{\frac{1}{2}}(l-2p+q)-(l-2p)]\sin\epsilon_{lmpq},$$

and $$\left(\frac{\mathrm{d}i}{\mathrm{d}t}\right)_{lmpq} = K_{lm}\frac{[(l-2p)\cos i-m]}{a(1-e^2)^{\frac{1}{2}}\sin i}[F_{lmp}(i)]^2[G_{lpq}(e)]^2\sin\epsilon_{lmpq}, \tag{21a}$$

with $$K_{lm} = \frac{Gm_☾ k_l}{[G(M+m_☾)\,a]^{\frac{1}{2}}}\left(\frac{R}{a}\right)^{2l+1}\frac{(l-m)!}{(l+m)!}(2-\delta_{0m}).$$

Also, with $n^2a^3=G(M+m_☾)$, $$\dot{n}_{lmpq} = -\frac{3}{2}\frac{n_{lmpq}}{a_{lmpq}}\dot{a}_{lmpq}. \tag{21b}$$

Long period tidal perturbations in the Moon's motion will arise when

$$(l-2p+q)-(l-2j+g) = 0,$$

for example for $pqjg = 0(-1)\,11,\ 1(-1)\,21$ etc. but the magnitudes of these perturbations are several times smaller than the secular effects. Table 2 summarizes the principal secular contributions to $\dot{a}$, $\dot{e}$ and $\mathrm{d}i/\mathrm{d}t$ for constant lag angle ϵ_{lmpq}. The principal contribution to $\dot{a}$ comes from the M_2 tide ($lmpq=2200$), for $\dot{e}$ the N_2 tide (2201) and for $\mathrm{d}i/\mathrm{d}t$ three tides M_2, O_1 (2100) and K_1 (2110) contribute about equally, with the last two almost cancelling each other. The relative importance of the perturbations with different frequencies, varies with the orbital elements and therefore with time and in general the diurnal tides become proportionally more important the further back into time the lunar motion is extrapolated.

In the actual integration of equations (21) it is usually more convenient to transform the elements into ecliptic variables such as the Hill-Brown variables which vary more linearly with time than do the equatorial elements, thereby facilitating the integrations. A complete analysis of the tidal evolution problem requires also a development for the solar tide and this is obtained from (19) by replacing all lunar parameters with their solar equivalents. The Sun–Moon interactions with each other and with the Earth's oblateness should also be considered, as should tides raised on the Moon by the Earth. The integrations are carried out by various assumptions about the lag angle ϵ_{lmpq}. If the magnitude and frequency dependence of the phase lag is known, then the effects of tidal dissipation are modelled by assigning a phase lag to each component in the tide expansion. Until recently it did not appear that the available tidal observations could permit this, even for estimating the present rate of change in the lunar orbit, but as discussed later, some progress in this area has been made. The usual procedure, following Darwin, has been to set the phase lags ϵ_{lmpq} equal for all frequencies. This would appear reasonable if the dissipation of tidal energy occurs within the solid Earth. Several frequency-dependent mechanisms appear to contribute to the dissipation of seismic waves within the mantle and for each, the specific dissipation function varies with depth due to pressure, temperature and compositional effects. The combined effect is apparently to produce a mantle with a broad frequency-independent seismic absorption band which encompasses periods from a few seconds to an hour or so. Whether or not this broad absorption peak also encompasses the tidal frequencies is unknown at present and one of the objectives of the tidal studies is to investigate this possibility. As demonstrated by Lambeck (1975), however, the major part of the dissipation occurs within the world's oceans and the phase angles now may be expected to be frequency-dependent. This is discussed below.

Table 2. Percentage contribution to the total secular changes in semi-major axis, eccentricity and inclination for constant phase lags, ϵ_{lmpq}

tide	*lmpq*	$\dot{a}$	$\dot{e}$	$\frac{di}{dt}$
M_2	2200	80.3	−7.5	81.2
N_2	2201	4.5	91.0	3.0
L_2	2201−1	—	−1.9	—
$2N_2$	2202	—	3.2	—
K_2	2210	—	—	7.7
O_1	2100	14.3	—	−70.3
Q_1	2101	0.8	16.2	−2.6
K_1	2110	—	—	79.3
Others	—	0.1	−1.0	1.7

Integration of the equations of the Moon's motion back into geological time were first carried out by Darwin (1908). More recent solutions have been attempted by Gerstenkorn (1955), Slichter (1963), MacDonald (1964), Sorokin (1965) and Goldreich (1966). These studies all agree in that (i) there has been a minimum Earth–Moon distance in the past, (ii) the inclination of the lunar orbit on the equator was substantially greater than it is now, and (iii) the eccentricity of the lunar orbit decreases as the distance increases. Differences do occur between these solutions and they have been discussed by Gerstenkorn (1967). The most complete integration of the Moon's motion is the work by Gerstenkorn (1955) and Goldreich (1966) although both have assumed that the lunar orbit has remained circular throughout and that dissipation in the Moon can be neglected. Gerstenkorn considers a phase lag that is proportional to frequency while Goldreich's results are for a constant ϵ_{lmpq} although he states that the evolution scenario is not significantly modified by a frequency dependent lag. Gerstenkorn's rationale for the linear phase lag–frequency relation is his adoption of a Maxwell Earth model although, as shown later, this also appears to approximate the dissipation in the oceans at the present time. Goldreich also considers the case treated by MacDonald and Slichter for constant geometric phase angles $\delta = (\dot{\theta}-n)\,\Delta t$. The latter implies that the time lag varies with the position of the Moon in its orbit and results in a complex dependence of the energy dissipation on frequency. This assumption appears to be of little consequence in the present tidal evolution but its consequences may have been important in the past if the Moon was ever close to the Earth (Gerstenkorn 1967). The most detailed study of the lunar motion during such a close approach eventuality, is the work of Gerstenkorn (1967). These theoretical developments of the dynamical orbital evolution can now be considered to be in a satisfactory state and what is much more problematical is the lack of a sound physical basis for extrapolating into the past: can we, from the presently inadequate understanding of the dissipating mechanism, assume that conditions in the past, particularly during the phase of close approach, have been constant? Gerstenkorn, Ruskol (1966) and others believe that the tidal dissipation occurs in the solid Earth and Moon, making the extrapolation into the past reasonably valid. But does this remain valid if, as discussed later, dissipation occurs mainly in the oceans?

Other than this time scale problem, the most significant result of the studies by Gerstenkorn and Goldreich is that the inclination of the lunar orbit on the equator must be non-zero when the Moon was at 10 Earth radii and that the Moon could never have moved in an equatorial orbit. This would appear to rule out theories in which the Moon is formed in the Earth's equator as is required by the fission and precipitation advocates of lunar origin (for example, Ringwood 1970; Binder 1974; Ringwood & Green 1974; O'Keefe 1974). Reviews by Kaula (1971) and Kaula & Harris (1975) discuss these consequences.

The dissipation implied by the lag angle ϵ_{lmpq} can also be described in terms of the specific dissipation Q^{-1}, or internal friction of the Earth through the relation

$$\tan \Phi = Q^{-1} = \Delta E / 2\pi E$$

where Φ is the phase of the strain behind the stress, that is the angle ϵ_{lmpq} (equation (19*b*)). ΔE is the energy dissipated per cycle and E is the peak energy stored. The astronomical estimate of $\dot{n}_{☾}$ is $-28''$ cy^{-2} (§3), and for constant ϵ_{lmpq} some 80% of this amount is due to the M_2 tide. With equations (21) this leads to a phase lag of $\epsilon_{2200} \approx 5°$ and the tidal effective Q of the Earth is about 12. With (18) the delay in the Earth's response to the M_2 tide-raising potential is of the order of 10 min. Integrating the three equations (21) with a constant phase lag $\epsilon_{lmpq} = 5°$, results in the Moon being within 10 Earth radii of the Earth about 1.15×10^9 years ago. For half this lag angle, this event would have occurred about 3.0×10^9 years ago.

Ocean tide potential

The above development is particularly relevant if the dissipation occurs in the solid Earth when the tidal bulge is harmonic in the same degree and order as the tide-raising potential and it has usually been supposed that severe local dissipation in the oceans may make this model a very poor fit to reality. As shown by Lambeck (1975), this is not the case and this is also implied by Kaula's (1969) equations. The Love number k_2 and the phase lag ϵ_{lmpq} entering into the potential (19) are tidal effective parameters in that they should reflect the total response of the Earth and its fluid layers to the gravitational attraction. These parameters are therefore not immediately comparable with other estimates of the Love numbers, either of observational or theoretical origin. In particular, because the resonance frequencies of some oceans are near the frequencies of some of the forcing functions (see for example, Platzman 1975), the tidal effective k_2 and phase lag ϵ_{lmpq} must be considered frequency-dependent and this suggests that it may be more appropriate to develop the ocean tide effect independently of the solid tide. This has been done by Lambeck, Cazenave and Balmino (1974) for the ocean tide perturbations in the orbits of close satellites and applied to the lunar orbit by Lambeck (1975).

The ocean tide component, β, is given at any position on the Earth by an amplitude $\xi^0_\beta(\varphi, \lambda)$ and a phase $\psi_\beta(\varphi, \lambda)$ that both vary with position. Thus (Hendershott & Munk 1970),

$$\xi_\beta(\varphi, \lambda, T) = \xi^0_\beta(\varphi, \lambda)\cos[2\pi f_\beta T - \psi_\beta(\varphi, \lambda)]. \tag{22}$$

The frequency of the tide f_β is traditionally expressed as a function of ecliptic coordinates rather than of the equatorial coordinates discussed earlier. A more uniform treatment would be to derive all the tidal perturbations in the ecliptic reference frame, but as we are concerned here only with the present rates of the tidal evolution and not with the integration of the equations of motion back into time, this has not been done. The phase ψ_β in (22) is expressed with respect to the Greenwich meridian and the time T is the mean solar time.

To obtain a global representation of the ocean tide in spherical harmonics that is convenient for an orbital theory, we expand $\xi^0_\beta \cos\psi_\beta$ and $\xi^0_\beta \sin\psi_\beta$ as follows:

$$\left.\begin{aligned} \xi^0_\beta \cos\psi_\beta &= \sum_{s=1}^{\infty}\sum_{t=0}^{s}(a'_{\beta,st}\cos t\lambda + b'_{\beta,st}\sin t\lambda)\,P_{st}(\sin\varphi),\\ \xi^0_\beta \sin\psi_\beta &= \sum_{s=1}^{\infty}\sum_{t=0}^{s}(a''_{\beta,st}\cos t\lambda + b''_{\beta,st}\sin t\lambda)\,P_{st}(\sin\varphi).\end{aligned}\right\} \tag{23a}$$

On the continents $\xi^0_\beta = 0$. Then

$$\xi_\beta = \sum_s\sum_t\sum_{+}^{-} D^{\pm}_{\beta,st}\cos[2\pi f_\beta T \pm t\lambda - \epsilon^{\pm}_{\beta,st}]\,P_{st}(\sin\varphi), \tag{23b}$$

with

$$\left.\begin{aligned} D^{\pm}_{\beta,st}\cos\epsilon^{\pm}_{\beta,st} &= \tfrac{1}{2}(a'_{\beta,st} \mp b''_{\beta,st}),\\ D^{\pm}_{\beta,st}\sin\epsilon^{\pm}_{\beta,st} &= \tfrac{1}{2}(a''_{\beta,st} \pm b'_{\beta,st}).\end{aligned}\right\} \tag{23c}$$

The summation of the form $\sum_{+}^{-} D^{\pm}\cos(\alpha \pm \beta - \epsilon^{\pm})$ implies $D^{+}\cos(\alpha + \beta - \epsilon^{+}) + D^{-}\cos(\alpha - \beta - \epsilon^{-})$.

The potential of this layer outside the Earth is

$$\Delta U_\beta(r) = 4\pi G R \rho_w \sum_s\sum_t\sum_{+}^{-}\frac{1+k'_s}{2s+1}\left(\frac{R}{r}\right)^{s+1} D^{\pm}_{\beta,st}\cos(2\pi f_\beta T \pm t\lambda - \epsilon^{\pm}_{\beta,st})\,P_{st}(\sin\varphi), \tag{24}$$

where the factor $1+k'_s$ allows for the Earth's elastic yielding under the variable ocean load. Expressing the coordinates r, φ, λ in Keplerian elements κ_i using the transformation (16) gives

$$\begin{aligned}\Delta U_\beta = \frac{4\pi G R^2}{a}\rho_w \sum_{s=1}^{\infty}\sum_{t=0}^{s}\sum_{+}^{-}\sum_{u=0}^{s}\sum_{v=-\infty}^{+\infty} \\ \times \frac{1+k'_s}{2s+1}\left(\frac{R}{a}\right)^{s} D^{\pm}_{\beta,st}F_{stu}(i)\,G_{suv}(e)\begin{bmatrix}\cos\\ \pm\sin\end{bmatrix}^{s-t\text{ even}}_{s-t\text{ odd}}\gamma^{\pm}_{\beta,stuv},\end{aligned} \tag{25}$$

with

$$\gamma^{\pm}_{\beta,stuv} = 2\pi f_\beta T - \epsilon^{\pm}_{\beta,st} \pm v_{stuv}.$$

The lunar (or solar) coordinates enter implicitly through the frequency f_β and through the amplitudes $D^{\pm}_{\beta,st}$. This expression for the ocean tide expansion can be substituted directly into the Lagrangian equations of motion to determine the perturbations in the orbit of a close Earth satellite whose motion is defined by the elements a, e, i, ω, Ω, M. Thus analogously to the solid tide perturbations (20), the ocean tide perturbation in inclination, for example, becomes

$$\begin{aligned}\left.\frac{\mathrm{d}i}{\mathrm{d}t}\right|_{\beta,stuv} = \frac{4\pi G R^2\rho_w}{a}\frac{1+k'_s}{2s+1}\left(\frac{R}{a}\right)^{s}\frac{1}{na^2(1-e^2)^{\frac{1}{2}}\sin i} \\ \times D^{\pm}_{\beta,st}F_{stu}(i)\,G_{suv}(e)\,[(s-2u)\cos i - t]\begin{bmatrix}\mp\sin\\ \cos\end{bmatrix}^{s-t\text{ even}}_{s-t\text{ odd}}\gamma^{\pm}_{\beta,stuv}.\end{aligned} \tag{26}$$

Long period perturbations, longer than one day, occur only when $\gamma^{\pm}_{\beta,stuv}$ does not contain the sidereal angle θ. As the argument $2\pi f_\beta T$ can be written as $m\theta + 2\pi f'_\beta T$ (table 3) where $m = 2$ for semi-diurnal tides, $m = 1$ for diurnal tides and $m = 0$ for zonal tides, diurnal and semi-diurnal tides will not give rise to long period perturbations for that part of the potential (25) containing $\gamma_{\overline{\beta},stuv}$. Only those coefficients $D^{+}_{\beta,st}$ of the semi-diurnal tide ($m = 2$) and $s, t = 2, 2; 4, 2; 6, 2; \ldots$ will give rise to long period terms. Other long period terms will be caused by the coefficients $D^{+}_{\beta,st}$ with $m = 2$ and $s, t = 3, 2; 5, 2; 7, 2; \ldots$, but now $v = \pm 1$ and the amplitudes of these terms will be smaller than those of previous coefficients by a factor e. Thus unless the satellite orbit is very eccentric, these perturbations will be quite small. Only coefficients $D^{+}_{\beta,st}$ with $s, t = 2, 1$; $4, 1$; $6, 1$; ..., of the diurnal tides ($m = 1$) will give rise to long period perturbations with $v = 0$. We note that the amplitudes of the perturbations are proportional to $(R/a)^{s+1}$ so that the coefficients with $s > 4$ will also tend to be small. Furthermore, we note that the perturbations due to $D^{+}_{\beta,21}$ or $D^{+}_{\beta,22}$ have the same dependence on the orbital elements as the perturbations due to the solid tide of the same frequency f_β and that the two cannot be separated. The $D^{+}_{\beta,42}$ occurs only in the ocean tide potential. Because of its different inclination function $F_{stu}(i)$, it can be separated from the leading term in the ocean tide expansion, even though the two have the same frequency, if at least two elements or two different orbits of close Earth satellites are available for analysis.

For the lunar orbit, the only terms in the potential (25) that are of importance are those with $s, t = l, m$ i.e. those for which the degree and order in the tide expansion are equal to the degree and order of the perturbing potential. The terms with $s = 2l$ which do give rise to secular perturbations, have a negligible effect on the present orbit because of the $(R/a_{☾})^s$ term. Terms with $s = 3$ are also negligibly small, of order $e_{☾}(R/a_{☾})^s$ times smaller than those due to $s = 2$. It is conceivable that at some stage in the past, these terms may have been more important. In analogy with the expressions (21), the secular perturbations in a, e and i, dropping the subscripts, are

$$\left.\begin{aligned} \dot{a}_{\beta,stuv} &= 2K'_{\beta,stuv}(s-2u+v)\begin{bmatrix}\sin\\ \cos\end{bmatrix}^{s-t\text{ even}}_{s-t\text{ odd}} \epsilon^{+}_{\beta,st}, \\ \dot{e}_{\beta,stuv} &= K'_{\beta,stuv}\frac{(1-e^2)^{\frac{1}{2}}}{ae}\left[(1-e^2)^{\frac{1}{2}}(s-2u+v)-(s-2u)\right]\begin{bmatrix}\sin\\ \cos\end{bmatrix}^{s-t\text{ even}}_{s-t\text{ odd}} \epsilon^{+}_{\beta,st}, \\ \left.\frac{\mathrm{d}i}{\mathrm{d}t}\right|_{\beta,stuv} &= K'_{\beta,stuv}\frac{[(s-2u)\cos i - t]}{a\sin i(1-e^2)^{\frac{1}{2}}}\begin{bmatrix}\sin\\ \cos\end{bmatrix}^{s-t\text{ even}}_{s-t\text{ odd}} \epsilon^{+}_{\beta,st} \end{aligned}\right\} \tag{27}$$

with
$$K'_{\beta,stuv} = \frac{3GMF_{stu}(i)\,G_{suv}(e)}{R[G(M+m_{☾})\,a]^{\frac{1}{2}}}\,\frac{1+k'_s}{2s+1}\,\frac{\rho_w}{\overline{\rho}}\left(\frac{R}{a}\right)^s D^{+}_{\beta,st},$$

where $\overline{\rho}$ is the mean density of the Earth.

Existing developments and computer programs of the evolution of the lunar orbit usually use the solid tide approach as discussed earlier and for this reason, it may be useful to determine equivalent Love numbers and phase lags that include the ocean effects. This is readily possible for the lunar orbit since the terms of degree $s = 2l$ in the ocean expansion are of no consequence. For close Earth satellite orbits this approach has little to offer. The equivalent phase lags are obtained by equating the secular perturbations in a given element due to the solid tide with those due to the ocean tide for the same frequency. The result is

$$\epsilon_{lmpq} = \frac{3M}{m_{☾}}\frac{\rho_w}{\overline{\rho}}\frac{1+k'_l}{k_l}\frac{1}{2l+1}\left(\frac{a}{R}\right)^{l+1}\frac{1}{R}D^{+}_{\beta,lm} \times \frac{(l+m)!}{(l-m)!\,(2-\delta_{0m})}\,\frac{1}{F_{lmp}(i)\,G_{lpq}(e)}\begin{bmatrix}\sin\\ \cos\end{bmatrix}^{l-m\text{ even}}_{l-m\text{ odd}} \epsilon^{+}_{\beta,lm}. \tag{28a}$$

Results for the principal tides are given in table 6 in the form

$$\sin \epsilon_{lmpq} = \chi_{lmpq} D^{+}_{\beta, lm} \begin{bmatrix} \sin \\ \cos \end{bmatrix}^{l-m \text{ even}}_{l-m \text{ odd}} \epsilon^{+}_{\beta, lm}. \tag{28b}$$

This expression would suggest that the equivalent lag angle ϵ_{lmpq} will change as the orbit evolves, but this is of course incorrect as the coefficients D^{+}_{lm} ϵ^{+}_{lm} themselves depend on the orbit. If we develop the equilibrium tide ${}^{\mathrm{E}}\xi$ in an equatorial system we find, following Lambeck, Cazenave & Balmino (1974) and Cazenave, Daillet & Lambeck (1977)

$$\begin{aligned} {}^{\mathrm{E}}\xi_{lmpq}(\varphi, \lambda; T) &= \frac{1}{2} \frac{(1+k-h)}{g} \frac{Gm_{☾}}{a} \left(\frac{R}{a}\right)^{l} (2-\delta_{0m}) \frac{(l-m)!}{(l+m)!} F_{lmp}(i_{☾})\, G_{lpq}(e_{☾}) \sum_i \sum_j \sum_u \sum_{\pm} Q^{\pm}_{lmiju} \\ &\quad \times P_{u|m\pm j|}(\sin\varphi) \left[a_{ij} \begin{pmatrix} \cos \\ \sin \end{pmatrix} \pm b_{ij} \begin{pmatrix} -\sin \\ \cos \end{pmatrix} \right]^{l-m \text{ even}}_{l-m \text{ odd}} [v_{lmpq} - (m \pm j)\,\lambda] \\ &= A_{\beta} \sum_i \sum_j \sum_u \sum_{\pm} Q^{\pm}_{lmiju} P_{u,|m\pm j|}(\sin\varphi) \left[a_{ij} \begin{pmatrix} \cos \\ \sin \end{pmatrix} \pm b_{ij} \begin{pmatrix} -\sin \\ \cos \end{pmatrix} \right]^{l-m \text{ even}}_{l-m \text{ odd}} [v_{lmpq} - (m \pm j)\lambda], \end{aligned}$$

where a_{ij} and b_{ij} are the coefficients in the cosine and sine terms in the spherical harmonic expansion of the ocean function and the $Q^{\pm}_{lmiju}$ are coefficients which are non-zero for $l+u+i$ even and for $\max(m \pm j, |l-i|) < u < l+i$. Comparison with (23) indicates that the quantity that can be expected to be constant during the orbit evolution, apart from physical changes in the sea floor configuration and in the volume of the oceans, is

$$\left(\frac{a}{R}\right)^{l+1} \frac{D^{+}_{\beta, lm}}{F_{lmp}(i)\, G_{lpq}(e)} \begin{bmatrix} \sin \\ \cos \end{bmatrix}^{l-m \text{ even}}_{l-m \text{ odd}} \epsilon^{+}_{\beta, lm}.$$

The above analysis indicates that the important effect of the ocean tide on the lunar orbit is given by one coefficient in the ocean expansion that is of the same degree l and order m as the lunar gravitational potential irrespective of the actual form of the ocean tide. For close Earth satellites this same coefficient gives rise to the dominant orbital perturbations but now, because of the nearness of the satellite to the ocean surface, a second term of degree $2l$ and order m in the tide expansion may become important. In both cases the other terms in the ocean expansion cause only periodic perturbations in the orbits with periods less than a day and are of no consequence; in the case of the Moon they do not lead to a permanent transfer of angular momentum from the Earth's spin to the orbital motion. Thus contrary to what is sometimes thought (see, for example, Goldreich & Peale 1968; Hipkin 1975), the simple time lagged ellipsoidal bulge is an adequate representation of the ocean tide. For the same reason Goldreich's (1966) criticism is invalid when he argues that MacDonald's (1964) and Kaula's (1964) developments provide a very poor fit to reality when severe local dissipation occurs: the development remains valid no matter how localized the actual dissipation may be; the satellite or the Moon will always respond to only the second degree harmonics in the tidal expansion. Severe localized dissipation will at most modify the amplitude and lag of this bulge.

Moon tides

Tides will also be raised on the Moon by the Earth's attraction, and the external potential of this tide follows from equation (19) by interchanging Earth and Moon parameters. This gives the potential per unit mass of the Earth and, in considering the changes in the orbital elements of the Moon, will have to be multiplied by $M/m_☾$. Substitution of this potential into the Lagrange equations results in the appropriate changes in the lunar orbit and complete expressions are given by Kaula (1964). For a given *lmpq*, the ratio of perturbations in *a*, *e*, *i* due to the Moon and Earth tides is given by, with $l = 2$,

$$\frac{\text{Moon tide effect}}{\text{Earth tide effect}} = \left(\frac{M}{m_☾}\right)^2 \left(\frac{R_☾}{R}\right)^5 \frac{(k_2 \sin \epsilon_{2mpq})_☾}{(k_2 \sin \epsilon_{2mpq})}.$$

For a homogeneous incompressible planet the Love number k_2 is given by (Jeffreys 1962)

$$k_2 = \frac{3}{2}\bigg/\left(1 + \frac{19\mu}{2\rho g R}\right),$$

where μ is the mean rigidity of the body. For the Moon, $k_2 \approx 0.02$. Data from the Apollo seismic network indicate a Q, for shear waves for the upper 500 or 600 km of the Moon, higher than several thousand, and below this depth a Q of about 300 with a central region Q of about 100 (Toksöz, Dainty, Solomon & Anderson 1974). If we take a lower limit of $Q \approx 150$ then $\epsilon_{lmpq} < 1°$ as compared with an observed tidal effective lag of about 5° for the Earth. Hence

$$\frac{\text{Moon tide effect}}{\text{Earth tide effect}} \lesssim 0.10.$$

The Moon tide equivalent to M_2 on the Earth ($lmpq = 2200$) is a permanent deformation and does not contribute to the dissipation. Hence tides raised on the Moon do not contribute significantly to changes in the semi-major axis of the lunar orbit. The contribution of Moon tides to $\mathrm{d}i/\mathrm{d}t$ is also relatively small but the contributions to the eccentricity of the lunar orbit may be relatively important since the principal effects on eccentricity result from the ellipticity of the lunar orbit or from the N_2 component.

Solar tides

Solar tides raised on the Earth will give imperceptible perturbations in the Earth's orbital motion and these can be estimated from the expressions for the Moon tide perturbations upon substitution of the Sun for the Earth and the Earth for the Moon; that is by substituting solar parameters for the lunar constants in (27) and multiplying by $m_\odot/M$.

Apart from the ocean tide, the atmospheric tide must also be considered and the most important contribution comes from the solar S_2 tide. The perturbations in the Earth orbit follow from expressions such as (27) in which the $D^+_{22} \sin \epsilon^+_{22}$ refers to the equivalent atmospheric layer (Lambeck *et al.* 1974). Already known to Kelvin (1890), this tide leads the Sun and tends to accelerate the Earth (see also Holmberg 1952).

Notation

In most tidal literature the definition of the tide is not given, in particular the sign of the lag is not specified and is as often positive as negative. This, in addition to conventions introduced by tidalists and astronomers which are not the same as conventions in satellite orbit theories, lead to complexities in the definition of the phase angle ϵ_{lmpq} and have introduced errors and misunderstanding in some of our earlier papers. The tide, as written in (22), uses as argument certain angles that involve time and lunar and solar elements as well as constants. The time is taken as Universal time and the orbital elements referred to the ecliptic are the s, h, p, N, p' defined as follows:

$$\begin{aligned} s(T) &= \text{mean longitude of the Moon } (\dot{s} = 0.549^\circ/\text{mean solar hour}),\\ h(T) &= \text{mean longitude of the Sun } (\dot{h} = 0.046^\circ/\text{h}),\\ p(T) &= \text{mean longitude of lunar perigee } (\dot{p} = 0.0046^\circ/\text{h}),\\ N(T) &= \text{mean longitude of ascending node } (\dot{N} = -0.0022^\circ/\text{h}),\\ p'(T) &= \text{mean longitude of perigee } (\dot{p}' = 0.000002^\circ/\text{h}),\end{aligned}$$

with respect to 0h.00, 1 January 1900.† Their initial values, rates and eventually quadratic and cubic terms are given by Doodson (1921). The constants in the arguments intervene because each harmonic component of the tide is written as a cosine of an argument and has positive amplitude, as has been done by Doodson & Warburg (1941) and presupposed in equation (22).

TABLE 3. RELATION BETWEEN THE INDICES AND ARGUMENTS, v_{lmpq} USED IN THE ORBITAL DEVELOPMENT AND THE ARGUMENTS, $2\pi f_\beta T$ USED IN THE OCEAN TIDE EXPANSION; r_β IS THE INTEGER REQUIRED TO MODIFY THE PHASE, $\epsilon^{+}_{\beta,st}$ DEFINED BY (23c)

tide	$lmpq$	origin L = Lunar S = Solar	$2\pi f_\beta T$	$v_{lmpq}+2\pi f_\beta T$	r_β
M_2	2200	L	$30^\circ T+2h-2s$	—	—
S_2	2200	S	$30^\circ T$	—	—
N_2	2201	L	$30^\circ T+2h-3s+p$	—	—
K_2	2210	L+S	$30^\circ T+2h$	—	—
L_2	220–1	L	$30^\circ T+2h-s-p+\pi$	$+\pi$	2
T_2	2201	S	$30^\circ T-h+p'$	—	—
$2N_2$	2202	L	$30^\circ T+2h-4s+2p$	—	—
K_1	2110	L+S	$15^\circ T+h+\frac{1}{2}\pi$	$+\frac{1}{2}\pi$	1
O_1	2100	L	$15^\circ T+h-2s-\frac{1}{2}\pi$	$-\frac{1}{2}\pi$	−1
P_1	2100	S	$15^\circ T-h-\frac{1}{2}\pi$	$-\frac{1}{2}\pi$	−1
Q_1	2101	L	$15^\circ T+h-3s-p-\frac{1}{2}\pi$	$-\frac{1}{2}\pi$	−1

Table 3 gives the complete arguments for the principal constituents. These correspond to the $2\pi f_\beta$ introduced in (22) and used in the potential development (25). For the lunar orbit we need only those terms in the perturbation equations such as (26), that give constant arguments. This requires that an equivalence be established between the tidal frequencies in the ecliptic reference frame and the equatorial frame in which the equations of motion have been expressed. An approximate equivalence is given by Lambeck, Cazenave & Balmino (1973) which is valid for the principal terms in the potential. Each harmonic of frequency f_β corresponds to a specific combination of indices $lmpq$ which define the v_{lmpq} and table 3 summarizes these relations. Furthermore the transformation (16) implies that the sidereal angle is zero when the mean Sun passes through the meridian of Greenwich but the definition of the Universal time intervening in $2\pi f_\beta$ is with respect to midnight. Thus a constant $m\pi$ must be introduced into the equivalence between $2\pi f_\beta$ and v_{lmpq}. The condition $\dot{\gamma}^{+}_{\beta,stuv} = 2\pi f_\beta - \dot{v}_{stuv} = 0$ (equation (25)) required for the secular perturbations due to the ocean tides gives

$$\gamma^{+}_{\beta,stuv} = \tfrac{1}{2}\pi r_\beta - \epsilon^{+}_{\beta,st} + m\pi,$$

† In astronomical discussions the time is usually defined with respect to 12 h.00, 31 Dec. 1899 or 12 h.00, 1 January 1900. Tidalists usually use the origin given here and used in table 3 (Bartels 1957).

where r_β is an integer and depends upon the harmonic considered (table 3). Then with the lag angle $\epsilon^{+}_{\beta,st}$ defined by equation (23) the lag angle appearing in the equations for the secular evolution of the orbit (27) should be replaced by

$$\epsilon^{+}_{\beta,st}-\tfrac{1}{2}\pi r_\beta-m\pi.$$

This has been implicitly done in the subsequent discussions of the lag angles and in table 4 which summarizes the values of the principal lag angles.

To conform closer to the usual description of ocean tides the definition (23) has been changed from the earlier useage by Lambeck *et al.* (1974) and Lambeck (1975) and this results in different values for the lag angles $\epsilon^{+}_{\beta,st}$. A misinterpretation of Hendershott's (1972) definition of the M_2 tide in these earlier papers has been corrected, but in all these cases the products $D^{+}_{22}\sin\epsilon^{+}_{22}$ which govern the accelerations and dissipation have remained unchanged. Some earlier confusion about the definition of the diurnal tides has also been resolved in the present paper.

5. Fluid tide models

Ocean tides

From the preceding analysis we have seen that the discussion of tidal dissipation in the oceans centres around the second degree harmonics in the ocean tide expansion (23); in particular on that harmonic that has the same degree and order as the exciting potential. Gravitational torques of the Moon on all the other harmonics in the ocean tide are either very small as for the degree 4 terms or are rigorously zero. Evaluation of the dissipation in the oceans only requires these few terms in the ocean tide and these can be estimated either from the analysis of existing ocean tide models or from the periodic response of close Earth satellites to the ocean tide.

Knowledge of the world's ocean tides is rather sparse. Long records of tides exist for many parts of the shoreline and are extremely valuable for predicting the tides locally but these tides are most often influenced by the coastline geometry and by the shallow coastal sea floor where frictional forces will significantly modify the tides from their open sea behaviour. Thus the coastal tides that sometimes reach more than 5 m, such as in the Bay of Fundy, are very local effects and are not at all characteristic of the global ocean tides. The best observations on the open ocean tides comes from island stations, particularly from those islands that rise up steeply from the deep sea floor such as volcanic islands. All such island records show that the undisturbed ocean tide does not in general exceed more than a metre. The limited availability of unperturbed tidal stations means that the global tidal patterns cannot be established with reliability from measurements alone and one has to resort to theory for estimating the global tides. The recent development of bottom pressure gauges for measuring the tides in the open sea has led to important improvements in the knowledge of regional ocean tides, as demonstrated by Munk, Snodgrass & Wimbush (1970) and Filloux (1971) for tides off the Californian coast and D. E. Cartwright and colleagues for the North Atlantic, but the information is still too sparse to be useful for global models. Complementary reviews of ocean tides are given by Hendershott & Munk (1970) and by Hendershott (1973).

This lack of adequate observational data means that the prediction of open ocean tides is largely based on theory, on the solution of the Laplace tidal equations (see for example, Lamb

1932; Platzman 1971). Complete solutions are complex and must include at least the following factors:

(i) The land–sea distribution and the ocean depth.

(ii) Energy dissipation, usually assumed to occur only in shallow seas and along coast lines.

(iii) The Earth being elastic is itself subjected to a solid tide which will work on the sea floor and modify the ocean tide.

(iv) The ocean tide, representing a variable load will deform the Earth and further modify the ocean tide.

(v) The computed tidal patterns must be in agreement with the observed tides for those stations where the tides are observed free from local disturbing effects.

Further complications, as yet not evaluated, may arise from the generation and dissipation of baroclinic tides.

Numerical solutions for the global M_2 tide have been published by Bogdanov & Magarik (1967); Tiron, Sergeev & Michurin (1967), Pekeris & Accad (1969), Zahel (1970, 1976) and Hendershott (1972). Pekeris & Accad and Zahel (1970) have both attempted to solve the Laplace tide equations using only a knowledge of the ocean–continent distribution, bathymetry and the tide generating force. Both consider the tide on a rigid Earth. Pekeris & Accad (1969) introduce a frictional force that is proportional to the tidal velocity rather than to the more realistic square of this velocity if dissipation is by bottom friction. This constraint and their definition of the boundaries of the oceans as the 1000 m isobath results in dissipation that is fairly uniformly distributed along coast lines. The authors determine a global friction coefficient by adopting a value that results in the best agreement between computed and observed tides. Zahel (1970, 1976) introduces a bottom friction force which is proportional to the square of the velocity and also allows for dissipation by turbulent friction, using a constant eddy viscosity coefficient for the world's oceans. Zahel's 1976 model differs from this earlier model in that (i) it considers the solid tide, (ii) the computational grid size has been reduced from 4° to 1° and (iii) the eddy viscosity coefficient has been reduced. In all these models, flow is not permitted across the coastlines since specific dissipation mechanisms have been introduced. Bogdanov & Magarik (1967) and Hendershott (1972) solve the Laplace tidal equations by imposing specific tide values along the coastlines. No frictional forces are introduced and currents are allowed to flow across the boundaries. Thus dissipation is assumed to occur in the shallow seas although the actual mechanism is not specified. Implicit in these studies is that the coastal observations are representative of the nearby deep water tides. Tiron *et al.* (1967) have published models for M_2, S_2, K_1 and O_1 but these models differ considerably from the others. This is possibly a consequence of their treatment of the dissipation, in that they impose the condition that observed tides equal computed tides where the former are reliably known, and that flow normal to the coastlines vanishes in regions where the tide is unknown. We do not discuss these models further.

Only Hendershott (1972) and Zahel (1976) consider the interaction between the solid and ocean tide although models such as that of Bogdanov & Magarik apparently give realistic results due to their introduction of observed tides as constraints. Allowing for the Earth's tidal deformation effectively reduces the tidal potential ΔU_2 to $(1+k_2-h_2)\,\Delta U_2$ and presumably the tidal amplitudes will be reduced by the factor $(1+k_2-h_2)$. But if dissipation is explicitly allowed for in the models, through mechanisms that depend on the tidal velocities, the rate of dissipation and hence the tidal amplitudes may be further modified. By an iterative procedure, Hendershott evaluates the interaction of the ocean floor's elastic yielding, under the variable tide load. His resulting model is inconclusive but does indicate that this interaction may be at least as important as the solid tide effect.† Presumably, by imposing the boundary conditions, this effect is already partly included in his and Bogdanov & Magarik's models.

Hendershott (1973) compares the various solutions. Agreement is generally good in the North Atlantic but elsewhere considerable discrepancies exist; in the Indian and Pacific oceans differences in amplitude by a factor of two occur between the solution of Pekeris & Accad and that of Hendershott, and we cannot consider the available models to be adequate as far as the

Table 4. Estimates of the $D^+_{22} \sin \epsilon^+_{22}$ or $D^+_{21} \cos \epsilon_{21}$ for different tide models

	tide		D^+_{22} or D^+_{21} cm	ϵ^+_{22} or ϵ^+_{21}	$D^+_{22} \sin \epsilon^+_{22}$ or $D^+_{21} \cos \epsilon^+_{21}$	$D^+_{22} \sin \epsilon^+_{22}$ or $D^+_{21} \cos \epsilon^+_{21}$ (corrected)
1.	M_2	Bogdanov & Magarik (1967)	4.33	126°	3.51	—
2.	M_2	Pekeris & Accad (1969)	4.57	110	4.37	3.00(1)
3.	M_2	Zahel (1970)	4.90	105	4.73	3.30(1)
4.	M_2	Hendershott (1972)	3.61	105	3.48	—
5.	M_2	Zahel (1976)	4.66	110	4.38	—
6.	M_2	equilibrium tide	4.00	0	—	—
7.	M_2	satellite solution(2)	3.07	123	2.6(2)	—
8.	M_2	satellite solution(2) + 1	—	—	3.0(2)	—
9.	S_2	Bogdanov & Magarik	1.87	140	1.20	1.42(3)
10.	S_2	satellite solution(2)	1.7	125	1.4(2)	—
11.	S_2	satellite solution(4)	1.5	121	1.3	—
12.	S_2	equilibrium tide	1.86	0	—	—
13.	S_2	atmospheric tide	0.34	292	−0.32(5)	—
14.	K_1	Dietrich (1944)	2.34	222	−1.74	—
15.	K_1	Bogdanov & Magarik (1969)	1.65	250	−0.56	—
16.	K_1	Zahel (1973)	6.64	221	−5.01	3.51(1)
17.	K_1	equilibrium tide	2.34	180	—	—
18.	O_1	Dietrich (1944)	1.66	38	1.31	—
19.	O_1	Bogdanov & Magarik (1969)	0.67	61	0.32	—
20.	O_1	equilibrium tide	1.66	0	—	—

(1) Reduced by factor $(1+k_2-h_2)$ to allow for yielding of Earth.
(2) Provisional results only (Daillet 1977).
(3) Corrected for atmospheric loading according to equation (30).
(4) Without correction for atmospheric tide.
(5) Without ocean response to atmospheric load.

detail is concerned. Available models for the other frequencies in the tidal potential must be considered even less adequate or are non-existent. Bogdanov & Magarik also published a solution for S_2 and in a later paper (Bogdanov & Magarik 1969) they give models for K_1 and O_1. Zahel (1973) gives a model for K_1. Empirical charts for the phase of the global K_1 and O_1 tides are given by Dietrich (1944). All of these models except those of Hendershott and Zahel have been harmonically analysed for the coefficients $D^{\pm}_{\beta,st}$ and $\epsilon^{\pm}_{\beta,st}$. At regular $10° \times 10°$ grid intervals the amplitudes $\xi^0_\beta(\varphi, \lambda)$ and phases $\psi_\beta(\varphi, \lambda)$ have been read off and the functions $\xi^0_\beta \cos \psi_\beta$ and $\xi^0_\beta \sin \psi_\beta$ expanded into spherical harmonics (equation 23*a*). On the continents, $\xi^0_\beta(\varphi, \lambda) = 0$. The coefficients a'_{st}, b'_{st}, a''_{st}, b''_{st} have been evaluated up to degree and order 18. Hendershott gives these coefficients directly. For Zahel's 1976 model his computed amplitudes and phases on a $1° \times 1°$ grid have been used directly. As is evident from the equations (27) expressing the secular rates in the Moon's orbital elements, only the second degree harmonics in the ocean tide expan-

† See also the recent models by Gordeev, Kagan & Polyakov (1977).

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sion enter into the discussion. Specifically we require $D^+_{22}\sin\epsilon^+_{22}$ for the semidiurnal tides, $D^+_{21}\cos\epsilon^+_{21}$ for the diurnal tides and eventually $D^+_{20}\sin\epsilon^+_{20}$ for the zonal tides. These coefficients are required for the principal tidal frequencies observed in the total ocean tide.

For the M_2 tide we have several ocean models which have been discussed above. The relevant coefficients in this study of tidal dissipation are summarized in table 4. The Pekeris & Accad (1969) and Zahel (1970) coefficients $D^+_{22}\sin\epsilon^+_{22}$ tend to be larger than those for the other M_2 solutions due, in part at least, to these models having been computed for a rigid Earth. The 'corrected' values in table 4 correspond to the tide coefficients on the rigid Earth reduced by the factor $1+k_2-h_2$, or by about 70%. The uncorrected amplitudes $D^+_{M_2,22}$ range from about 4.3 to 4.9 cm and the phase varies by about 20°. The agreement appears satisfactory despite the differences in the manner in which the dissipation is treated. The only other tide for which a comparison is possible is the K_1 tide for which we have the numerical models of Bogdanov & Magarik and of Zahel and the empirical model of Dietrich. For the last, only the phase is available but amplitudes are known along the coastlines. With these and the location of the amphidromic systems, we have extrapolated for the tide amplitudes in the open ocean with the condition that the $D^+_{K_1\,21}$ term has the same amplitude as the equilibrium tide term. This same process applied to Dietrich's model of the M_2 tide yields a satisfactory result. The comparison of these three K_1 models reveals the unsatisfactory status of the global diurnal tide models. Zahel's corrected amplitude $D^+_{K_1,21}$ is about three times greater than the other two values and may be a consequence of his using a value for the effective eddy viscosity that is too large (Zahel, private communication). All diurnal tides summarized in table 4 indicate that $|\cos\epsilon^+_{21}| < |\sin\epsilon^+_{22}|$ suggesting that the diurnal tides are closer to their equilibrium values than the semi-diurnal tides.

The coefficients ${}^{\mathrm{E}}D^+_{\beta,st}$ for the equilibrium tide are obtained by comparing the expressions (23*b*) and (29). As we are concerned with only secular effects on the lunar orbit we have $m = t$. We also require $u = s$ and $|m\pm j| = t$. Thus for $|m+j| = m, j = 0$ and for $|m-j| = 0, j = 0$ or $2m$. Also $l = 2$ and $s = 2$. Then with $\alpha = \frac{1}{2}\pi r_\beta + m\pi$

$$
\begin{aligned}
{}^{\mathrm{E}}D^{\pm}_{\beta,2m}\cos{}^{\mathrm{E}}\epsilon^{\pm}_{\beta,2m} = A_\beta\Bigg\{\sum_i Q^{\pm}_{2mi02}\,a_{i0}\begin{bmatrix}\cos\alpha\\ \sin\alpha\end{bmatrix}^{2-m\text{ even}}_{2-m\text{ odd}} \\
+ Q^{-}_{2mi(2m)2}\left[a_{i(2m)}\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix} - b_{i(2m)}\begin{pmatrix}-\sin\alpha\\ \cos\alpha\end{pmatrix}\right]^{2-m\text{ even}}_{2-m\text{ odd}}\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
{}^{\mathrm{E}}D^{\pm}_{\beta,2m}\sin{}^{\mathrm{E}}\epsilon^{\pm}_{\beta,2m} = A_\beta\Bigg\{\sum_i Q^{\pm}_{2mi02}\,a_{i0}\begin{bmatrix}\sin\alpha\\ -\cos\alpha\end{bmatrix}^{2-m\text{ even}}_{2-m\text{ odd}} \\
+ Q^{-}_{2mi(2m)2}\left[a_{i(2m)}\begin{pmatrix}\sin\alpha\\ -\cos\alpha\end{pmatrix} - b_{i(2m)}\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix}\right]^{2-m\text{ even}}_{2-m\text{ odd}}\Bigg\}.
\end{aligned}
$$

Atmospheric tide

In many discussions on the secular deceleration of the Earth's rotation, the atmospheric tide has played a rôle that is well beyond its significance in the total tidal effect; neglected ocean tides such as O_1, K_1 and the virtually unknown N_2, are much more important (Lambeck 1975). This apparent importance is undoubtedly related to the fact that the atmospheric tide leads the Sun and hence gives a positive acceleration to the Earth in contrast to the ocean's decelerating effect. This has led to speculation about the possible rôle it may have played in the past (Holmberg 1952) but at atmosphere similar to that of Venus, and thermally driven, is required if it is to cancel the effects of the present ocean tides.

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Chapman & Lindzen (1970) review the subject of atmospheric tides (see also Siebert 1961). Of the numerous frequencies in the atmospheric tide, the only one of some importance is the solar tide at the S_2 frequency. Unlike the ocean tides, its global distribution is reasonably well known from ground level pressure records and, because it is less influenced by the ocean–continent distribution, it is more uniformly distributed over the globe: in its harmonic expansion proportionally more power is contained in the low degree than high degree harmonics, and the ratio of the 2, 2 terms in the atmospheric and oceanic expansions will be larger than would be expected from simply comparing the tidal amplitudes at any one station.

The gravitational potential of the atmosphere can be computed in the same way as the ocean tide. Atmospheric pressure $p(\varphi, \lambda)$ at each point on the Earth's surface is harmonically analysed into its constituent frequency components $p_\beta(\varphi, \lambda)$. Each such component p_β can be expressed by a series analogous to the ocean tide expansion (13) and the surface load is p_β/g. In computing the potential outside the Earth due to the layer, the effective depth of the atmosphere is assumed to be small compared with the earth's radius. The elastic yielding of the Earth under this variable load is allowed for by introducing the load deformation coefficients k'_s. More troublesome is the question of how to allow for the ocean response to this load: does the ocean respond as an inverted barometer or not? A study by Cartwright (1968) of the radiational tides around the British Isles found an average ratio (radiational tide)/(gravitational tide) of 0.18 for S_2 and an average phase difference between these tides of 130°. Zetler (1971), in a similar study, for stations along the U.S. coast finds comparable results. This radiational tide appears to be largely excited by the atmospheric tidal load on the ocean surface but the response is very different from that of an inverted barometer: In some cases the radiational tide has an amplitude that is larger than the predicted amplitude by a factor of 10 (see also Cartwright & Edden 1977). This may be partially due to a direct effect of solar radiation on the sea surface or due to the offshore–onshore wind cycle. The first effect would not load the crust, the second effect will modify only the coastal tide, not the mid-ocean tides. If the ocean response is static, the correct treatment is to multiply the surface load p_β/g of the atmospheric tide by the continent function $[1-l(\varphi, \lambda)]$ and expressing the resulting product of two spherical harmonic expansions as a linear relation of spherical harmonics in which each term of degree s is multiplied by the appropriate $(1+k'_s)$. In view of the various uncertainties this treatment is probably unwarranted. Table 4 gives the relevant coefficient in the S_2 tide assuming that there is no ocean response. These are from a solution by W. Kertz, B. Haurwitz and A. Cowley as published by Chapman & Lindzen (1970).

The S_2 ocean tide solution of Bogdanov & Magarik (1967) may include the ocean response to the atmospheric tide since it has been constrained by tide observations along the coastlines and presumably no correction for this has been made. A 'corrected' S_2 ocean tide coefficient then would be

$$D^{+}_{22} \sin \epsilon^{+}_{52} = (D^{+}_{22} \sin \epsilon^{+}_{22})_{\text{observed ocean}} - a_{00}(D^{+}_{22} \sin \epsilon^{+}_{22})_{\text{atmosphere}}. \tag{30a}$$

Age of the tide

It has been known for a long time that spring tides do not occur as zysygy but some time later and that this delay is a consequence of different phase lags associated with the M_2 and S_2 tides (see for example, Lamb 1932; Doodson & Warburg 1941). Consider the M_2 and S_2 tides defined by (22). Their superposition can be expressed as

$$\bar{\xi} = \xi_{M_2} + \xi_{S_2} = \bar{\xi}^0 \cos(2\pi f_{M_2} T - \psi_{M_2} - \eta)$$

with amplitude

$$\bar{\xi}^0 = [(\xi^0_{M_2})^2 + (\xi^0_{S_2})^2 + 2\xi^0_{M_2}\xi^0_{S_2} \cos \nu]^{\frac{1}{2}}$$

and phase $$\eta = \arctan \frac{\xi^0_{S_2} \sin \nu}{\xi^0_{M_2} + \xi^0_{S_2} \cos \nu},$$

where $$\nu = 2\pi f_{M_2} T - 2\pi f_{S_2} T - (\psi_{M_2} - \psi_{S_2}).$$

The comparable amplitude for the superposition of the equilibrium M_2 and S_2 tides is

$$\bar{\xi}^0_E = \{(\xi^0_{M_2|E})^2 + (\xi^0_{S_2|E})^2 + 2\xi^0_{M_2|E}\, \xi^0_{S_2|E} \cos \nu_E\}^{\frac{1}{2}}$$

with $$\nu_E = 2\pi f_{M_2} T - 2\pi f_{S_2} T.$$

The maximum equilibrium tide occurs when

$$2\pi(f_{M_2} - f_{S_2})\, T = 2p\pi,$$

where p is an integer. The maximum observed tide occurs when

$$2\pi(f_{M_2} - f_{S_2})\, T - (\psi_{M_2} - \psi_{S_2}) = 2p\pi$$

and is late by $$(\psi_{M_2} - \psi_{S_2})/2\pi(f_{M_2} - f_{S_2}).$$

This is the 'age of the tide', a term introduced by W. Whewell in 1833. It is typically of the order of one day (Doodson & Warburg 1941) although it may vary regionally between ± 5 days. Garrett & Munk (1971) have estimated the age from 647 port records as 1.2 days. The age can also be expressed in terms of the $D^+_{\beta,st}$ and $\epsilon^+_{\beta,st}$ of the M_2 and S_2 tides by using the relations (23*a*) and (23*c*). The resulting expression is considerably simplified if the observed value does indeed represent a global mean and can be associated with the 2, 2 terms in the tide expansion. Furthermore, for the available M_2 and S_2 models, the $D^+_{\beta,22}$ are small compared with $D^-_{\beta,22}$ and we obtain

$$\psi_{M_2} - \psi_{S_2} \approx \epsilon^+_{M_2,22} - \epsilon^+_{S_2,22}.$$

Bogdanov & Magarik's solutions for M_2 and S_2 give

$$\epsilon^+_{M_2,22} - \epsilon^+_{S_2,22} = -14°$$

corresponding to an age of about 0.6 days. The satellite results (see below) for M_2 and S_2 give $-3°$ or an age of about 0.2 days. The differences between the three estimates could indicate (i) unsatisfactory models and satellite results, (ii) that the coastal values are not representative of mid-ocean values or (iii) that the S_2 values are contaminated by radiational tides. In view of the spread of the lag angles ϵ^+_{22} obtained from the various models, the first interpretation is reasonable. Also the M_2 satellite results remain unsatisfactory because of the poor separation of the 2, 2 and 4, 2 terms. To investigate the second possibility the spatial variation of the age of the tide should be investigated more closely. Contamination of S_2 by the pressure and other radiational tides is likely. The satellite results have been corrected for the former by assuming no ocean response to the variable pressure whereas the port results would include an ocean response.

Doodson & Warburg (1941) note that on the average N_2 and M_2 differ in phase by about 15°. Garrett & Munk (1971) refer to a result by M. Wimbush that the average phase lag of the N_2 tide with respect to M_2 is about one half of the lag of M_2 with respect to S_2. That is $\epsilon_{S_2} < \epsilon_{M_2} < \epsilon_{N_2}$ and $\epsilon_{S_2}: \epsilon_{M_2}: \epsilon_{N_2} \approx 4:2:1$. The variation of these angles with frequency is approximately linear and given by

$$\epsilon^+_{\beta,22} = \epsilon^+_{M_2,22} + \frac{\epsilon^+_{S_2,22} - \epsilon^+_{M_2,22}}{2\pi f_{S_2} - 2\pi f_{M_2}} (2\pi f_\beta - 2\pi f_{M_2}). \tag{31a}$$

In earlier studies (Lambeck 1975; Lambeck & Cazenave 1977) it was assumed that the phase angles for all semi-diurnal tides could be considered to be the same but the above result may be a better approximation. Thus for other semi-diurnal tides we assume that the lag is given by $\epsilon^+_{\beta, 22}$ and that

$$D^+_{\beta, 22} = D^+_{\mathrm{M}_2, 22} \frac{{}^{\mathrm{E}}D^+_{\beta, 22}}{{}^{\mathrm{E}}D^+_{\mathrm{M}_2, 22}}. \tag{31 b}$$

Dietrich (1944) tabulates the lag of the diurnal tide for some 100 coastal and island stations and the average value is $\psi_{\mathrm{K}_1} - \psi_{\mathrm{O}_1} \sim 2°$. Bogdanov & Magarik's solutions for these two tides yield 9°. These younger ages, compared with the semi-diurnal age, are further evidence that the global diurnal tides are further from resonance than the semi-diurnal tides.

6. Satellite results

The motions of close Earth satellites and of the Moon are both perturbed by the Earth and ocean tides but with the difference that for close satellites the principal perturbations are periodic while for the Moon the principal perturbations are secular. As the same coefficients in the tide expansions are responsible for both types of orbital disturbances, the analysis of satellite orbits provides parameters that can be directly applied to describe the secular tidal evolution of the Moon's motion (Lambeck 1975; Lambeck & Cazenave 1977). The immediate advantage of this approach is that the total tidal deformation is taken into account – not only the separate solid and fluid tides but also their interactions: the effect of the solid tide on the ocean tide, the yielding of the Earth under the variable ocean load and the yielding of the solid Earth and ocean surfaces under the variable atmospheric load. This advantage is particularly important while there is not yet an adequate way of treating these interactions in the present numerical solutions of the Laplace tide equations. Satellite results do not, however, provide any information on the nature of the energy sink.

Cazenave, Daillet & Lambeck (1977) have given initial results for the M_2 and S_2 tide parameters $D^+_{22} \sin \epsilon^+_{22}$ that enter into the lunar discussion. Their solution was based on variations in the inclination of two satellites: 6709201, a satellite of the TRANET Doppler navigation network and GEOS 1 (6508901). Since then, perturbations in the inclination and right ascension of the three other satellites tracked by the TRANET network have been analysed by using the orbital residuals provided by R. J. Anderle. Unfortunately the four Doppler satellites are in quite similar orbits and do not permit a separation of the second and fourth degree harmonics in the tide expansion (see Cazenave *et al.* 1977). Neither can the perturbations in right ascension be used as there is a second equally important tidal component, due to the O_1 tide, at a nearby frequency and the two cannot be separated from perturbation analyses alone. Because of the less uniform distribution of observations, the GEOS 1 satellite does not contribute significantly to the solution for the M_2 tide. Recently C. Goad and B. Douglas have provided us with elements for the GEOS 3 satellite (Goad & Douglas 1976) and these lead to an improved solution in that a separation of the second and fourth degree harmonics is now possible, although the correlation between the two is still high, of the order 0.8 (Daillet 1977).

This solution (table 4) as well as earlier ones suggest that the coefficient $D^+_{22} \sin \epsilon^+_{22}$ is less than the values obtained from the numerical models. This may be a consequence of the poor separation of the two terms or it may be an indication that the elastic yielding of the Earth by the ocean tide, a factor that has not been treated adequately in the numerical models now available, reduces the tidal amplitudes.

Until better solutions are available, probably the most satisfactory solution is to combine the satellite results with the numerical model results. This has been attempted by Cazenave *et al.*

(1977). The observation equations describing the ocean models are (23 *a*) with (23 *b*); the amplitudes and phases are given in a 10° × 10° grid. The variances of the amplitudes and phases are taken as

$$\sigma^2_{\xi^0}/\langle \xi^2 \rangle = \sigma^2_{\psi};\ \sigma_{\xi^0} \sim 5\,\text{cm} \quad (\sigma_{\psi} \approx 6°).$$

and we ignore any correlation between these two quantities as well as any correlation between these values at nearby point. We are forced to make these simplifications for lack of further information on the statistical behaviour of the numerical solutions. The $\xi^\circ \cos\psi$ and $\xi^\circ \sin\psi$ at any one point are of course correlated. The satellite observation equations are given by expressions such as (26). A combination between the satellite solution and the numerical model of Bogdanov & Magarik (1967) gives $D^+_{22} \sin \epsilon^+_{22} \approx 3.0\,\text{cm}$ and is within the error bounds placed on the coefficients in table 4 for the ocean tide results.

For S_2 the satellite solution is more satisfactory in view of the good separation that is possible between the degree 2 and 4 terms from the polar navigation satellites and GEOS 1, since the perturbations due to this tide have a period that is very much longer than that caused by the M_2 tide and very much longer than the sampling interval of the GEOS 1 orbit determinations. Residual solar radiation pressure perturbations with the same frequency as the S_2 tide may still contaminate the solution. More information from satellites in distinctly different orbits is essential before we can conclude that these discrepancies are real, and at present there is no reason for preferring the satellite results to the numerical model results. Additional satellite orbits must be analysed to give results for the tidal components other than these two. Felsentreger, Marsh & Agreen (1976) give linear relations between the degree 2 and 4 terms for the S_2, K_1 and P_1 tides but these are of no value in the lunar problem. In addition, their results for P_1 are erroneous since they have neglected the equally important K_2 tide which perturbs the orbit of GEOS 1 with a frequency that is very similar to that of the P_1 perturbation. Further analyses of these and other orbits are in progress (Daillet 1977).

7. Energy dissipation in the oceans

Dissipation equations

The energy equation relevant to the tidal oscillation is (see for example, Hendershott 1972)

$$\frac{\mathrm{d}}{\mathrm{d}t}(Z_\mathrm{k} + Z_\mathrm{p}) + \boldsymbol{\nabla}_\mathrm{s}.(g\boldsymbol{u}D\boldsymbol{\xi}') = \frac{\mathrm{d}E}{\mathrm{d}t} + \frac{\mathrm{d}W}{\mathrm{d}t},$$

Z_k and Z_p are the densities of kinetic and potential energies per unit of ocean surface; the second term on the left hand side represents the horizontal power flux associated with the horizontal tidal currents $\boldsymbol{u}$; ξ' is the geocentric tide; D the depth of ocean; $\mathrm{d}W/\mathrm{d}t$ is the rate at which work is done on the ocean and $\mathrm{d}E/\mathrm{d}t$ is the rate at which energy is dissipated, either by bottom friction or by some other unspecified mechanism. Integration of this equation over one tidal period P and further integration over the entire ocean surface reduces the energy equation to

$$-\langle \dot{E} \rangle = \int_\text{ocean} \langle \dot{W} \rangle\, \mathrm{d}S, \tag{32a}$$

with

$$\langle \dot{W} \rangle = \frac{1}{P}\int_{T=T_0}^{T=T_0+P} \dot{W}\, \mathrm{d}t, \tag{32b}$$

since the coastlines are assumed to be impermeable. Thus the rate at which energy is dissipated throughout the ocean is proportional to the rate at which work is done on it by the total tidal force.

If, as is commonly believed, either for want of contradictory information or for reasons of tradition, bottom friction in shallow seas is the dominant dissipating mechanism then

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \boldsymbol{u}\cdot\boldsymbol{F},$$

where $\boldsymbol{F}$ is the bottom stress vector and can be written as $\rho\alpha|\boldsymbol{u}|\,\boldsymbol{u}$ and α is the coefficient of friction. Taylor (1919) adopts $\alpha = 0.002$; Brettschneider (1967) proposes $\alpha = 0.003$. Dissipation will therefore be proportional to the cube of the tidal current velocity. In the open seas the solution of the Laplace tidal equations give tidal velocities that are typically of the order of $1\,\mathrm{cm\,s^{-1}}$ and with $\alpha \sim 0.002$, $\rho\alpha u^3 \approx 0.002\,\mathrm{erg\,s^{-1}\,cm^2}$. Integrating over the world's oceans gives $-\mathrm{d}E/\mathrm{d}t \approx 10^{16}\,\mathrm{erg\,s^{-1}}$ compared with the required amount of about $4\times10^{19}\,\mathrm{erg\,s^{-1}}$ if the astronomically observed tidal accelerations are a consequence of dissipation in the oceans. This is the usual argument to show that the dissipation is limited to very shallow seas where the observed tidal currents are much larger than those in the open sea. Even the coastal shelfs, where the water depth is of the order of 200 m, are usually thought to provide an inadequate energy sink (Jeffreys 1962; Munk 1968; but see Defant 1961, for a contrary view). Thus

$$-\left\langle\frac{\mathrm{d}E}{\mathrm{d}t}\right\rangle = \int_{S'}\int_{T_0}^{T_0+P} \rho_w\alpha|\boldsymbol{u}|\,\boldsymbol{u}\cdot\boldsymbol{u}\,\mathrm{d}t\,\mathrm{d}S', \tag{33}$$

where the integral is carried out over the shallow seas of area S'.

If this argument is accepted then the amount of energy that is dissipated can also be evaluated from the energy balance across the entrances to the shallow seas: That is, the rate of energy flux across the entrance during one cycle of the tide added to the rate of energy produced by the work done on the sea by the Moon, must equal the energy dissipated in the enclosed body of water. The energy flux across the boundary L includes the rate at which work is done by the water entering across the boundary and the kinetic and potential energies carried along by the tidal current although Jeffreys (1929) shows that only the rate at which work is done by the water is important. That is

$$-\left\langle\frac{\mathrm{d}E}{\mathrm{d}t}\right\rangle = g\rho_w\int_L\int_T Du\xi\,\mathrm{d}t\,\mathrm{d}L + \frac{\mathrm{d}W}{\mathrm{d}t}. \tag{34a}$$

Furthermore, the areas of the shallow seas are quite small and the work done by the Moon and Sun on them is considered to be small, as shown by Taylor's calculation for the Irish sea where he found a rate of change of energy flux of $6.4\times10^{17}\,\mathrm{erg\,s^{-1}}$ while the rate at which work was done by the Moon on the Irish sea was only $0.4\times10^{17}\,\mathrm{erg\,s^{-1}}$. The rate at which energy must be dissipated therefore reduces to

$$-\left\langle\frac{\mathrm{d}E}{\mathrm{d}t}\right\rangle = g\rho_w\int_E\int_T Du\xi\,\mathrm{d}t\,\mathrm{d}L, \tag{34b}$$

where the integral is evaluated across the entrances to the shallow seas. From Zahel's (1976) maps of the global distributions of energy dissipation by bottom friction and of the rate at which work is done on the ocean surface, it is not always evident that the rate of work done on the shallow sea can be ignored; for several regions $|\mathrm{d}W/\mathrm{d}t|$ is considerably larger than $\mathrm{d}E/\mathrm{d}t$ and $\mathrm{d}W/\mathrm{d}t$ may be positive or negative.

The integrals (32), (33) and (34) summarize the standard discussion of dissipation (see for example, Munk & MacDonald 1960; Kaula 1968) and has changed very little over the last fifty years. Application of all three methods has been beset by numerous difficulties. On reading the original accounts of these attempts, one is immediately struck by the paucity of relevant observations and by the hypotheses and extrapolations that have been made in order to arrive at a global estimate and it is surprising that there is any agreement at all between the various estimates.

Table 5. Estimates of rate of energy dissipation in the oceans

author	method	tide	$\mathrm{d}E/\mathrm{d}t$ 10^{19} erg s^{-1}	$\mathrm{d}E/\mathrm{d}t$ (corrected) 10^{19} erg s^{-1}
Jeffreys (1920)	bottom friction	M_2	1.1	—
Heiskanen (1921)	bottom friction	M_2	1.9(1)	—
Groves & Munk (1958)	torque	M_2	3.2	—
Miller (1966)	energy flux	M_2	1.7	—
Pekeris & Accad (1969)	bottom friction	M_2	6.0	4.20(2)
Hendershott (1972)	torque	M_2	3.0	—
Kuznetsov (1972)	torque (Zahel model)	M_2	7.28	3.57(3)
	(Pekeris & Accad model)	M_2	6.68	3.29(3)
Pariyskiy *et al.* (1972)	(Bogdanov & Magarik model)	M_2	5.24	3.67(2)
Zahel (1976)	bottom friction and turbulence	M_2	3.8	—
equation (40) this paper	Pekeris & Accad model	M_2	2.92(4)	—
	Zahel model (1970)	M_2	3.20(4)	—
	Hendershott model	M_2	3.70	—
	Bogdanov & Magarik model	M_2	3.23	—
	Zahel model (1976)	M_2	4.25	—

(1) Value given by Munk & MacDonald (1960).
(2) Reduced by a factor $(1+k-h)_2$ (see text).
(3) Reduced by a factor $(1+k-h)$ (see text).
(4) Based on the corrected $D_{22}^{+} \sin \epsilon_{22}^{+}$ terms in table 4.

The first method (32), the evaluation of the rate at which work is done by the Sun and Moon on the ocean surface, requires that the global ocean tide is known everywhere and it is often assumed that the present models are inadequate for this. The advantage of the method is that it requires no assumption about the nature of the energy sink in the oceans and neither does it depend upon a knowledge of the tidal currents. The method was first used by Heiskanen (1921) and later by Groves & Munk (1958). The most recent attempts at directly evaluating the integral (32) have been made by Pariyskiy, Kuznetsov & Kuznetsova (1972), Kuznetsov (1972) and Hendershott (1972) although the form of the ocean tide (23*b*) and its influence on the lunar orbit (equations 27) suggest that this calculation can be considerably simplified (see below).

The bottom friction method (integral 33) was first used by Taylor (1919) in his discussion of dissipation in the Irish sea and was extended to the world's oceans by Jeffreys (1920) and by Heiskanen (1921). Table 5 gives their global estimates for $\mathrm{d}E/\mathrm{d}t$. The value attributed to Heiskanen is that corrected by W. D. Lambert and is given by Munk & MacDonald (1960). Both values are only of historical interest today. Miller (1966) has used the energy flux method (integral 34*b*) which also assumes that dissipation is restricted to shallow seas but the actual mechanism need not be specified. As the energy flux is proportional to the first power of the tidal current velocity, it is generally considered to be more precise than the bottom friction method although it does require a knowledge of the tidal height across the entrances to the shallow seas, and of the time of maximum current relative to the time of the maximum tide amplitude. Miller ignores the $\mathrm{d}W/\mathrm{d}t$ term in (34*a*).

Dissipation by mechanisms other than bottom friction have generally received only a qualitative treatment (see the discussion in § 8) and have not been explicitly introduced into the global tidal models. An exception to this are the M_2 tide solutions by Zahel (1970, 1976), who introduces dissipation by turbulence in addition to bottom friction. Tidal currents can inject energy into horizontal eddies by several mechanisms: by lateral stresses set up along the coast or continental shelves, by bottom topography or by adjacent tidal currents. This suggests that the dissipation by turbulence may be most important along the continental margins. Turbulent motion and its dissipative action is a complex mechanism and cannot be readily taken into account in the tidal equations. Usually it is introduced qualitatively by introducing an effective viscosity or eddy viscous force and this has proved useful in providing simple dissipative mechanisms in a number of oceanic circulation problems. These eddy viscous forces are given by Zahel (1973, 1976) (see also Kasahara & Washington 1967) who assumes that vertical turbulence can be neglected in comparison to the lateral turbulence. The forces to be added to the Laplace tidal equations together with other frictional forces, then are

$$F_\lambda = \rho K_\mathrm{h} \nabla^2 u_\lambda,$$

$$F_\varphi = \rho K_\mathrm{h} \nabla^2 u_\varphi,$$

with the operator
$$\nabla^2 = \frac{1}{R^2}\left\{\frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial}{\partial\varphi}\right) + \frac{1}{\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2}\right\},$$

where u_λ and u_φ are the longitudinal and latitudinal components of the horizontal velocity and K_h is the lateral eddy viscosity coefficient. K_h depends on the type and scale of the turbulent motion as well as on the degree of stability and in any flow pattern will vary spatially and in time. Estimates of K_h vary over a wide range. Munk (1950) requires $K_\mathrm{h} \sim 5\times10^7\,\mathrm{cm^2\,s^{-1}}$ if the energy acquired by the ocean circulation from the winds is dissipated by lateral viscosity, while values up to $10^9\,\mathrm{cm^2\,s^{-1}}$ are required to account for features of the western boundary currents (Bowden 1962). For the Antarctic circumpolar current Hidaka & Tsuchiya (1953) estimate $K_\mathrm{h} \approx 10^{10}\,\mathrm{cm^2\,s^{-1}}$. Zahel (1970, 1973) adopts a constant value for the world's oceans of $10^{11}\,\mathrm{cm^2\,s^{-1}}$ but reduces this to 5×10^9 in his most recent model in which he finds

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\text{bottom friction}} = 0.7\times10^{19}\,\mathrm{erg\,s^{-1}},$$

and
$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\text{turbulent friction}} = 3.0\times10^{19}\,\mathrm{erg\,s^{-1}}.$$

This importance of turbulent friction may also be a consequence of his choice of boundary conditions; the coastline is defined by the 50 m depth contour and the velocity perpendicular to the coastline vanishes. Thus a major part of the shallow seas is excluded from his model. Gordeyev, Kagan & Rivkind (1975) adopts $K_\mathrm{h} \approx 10^7\,\mathrm{cm^2\,s^{-1}}$ and conclude that with this value dissipation by turbulence is not important. Clearly more precise information on a representative value of the eddy viscosity, applicable to tidal problems, is most desirable.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Work method

The rate at which work is done on the ocean consists of two parts: (i) the rate at which body forces work on the ocean,
$$\rho\psi\frac{\mathrm{d}}{\mathrm{d}t}(\xi'-\delta)+\rho\boldsymbol{\nabla}_s.[\boldsymbol{u}(\xi'-\delta+D)\,\psi],$$
and (ii) the rate at which the sea floor, moving due to the solid tide and the variable ocean load, works on the ocean, or
$$\rho g(\xi'-\delta+D)\,\mathrm{d}\delta/\mathrm{d}t.$$
In these expressions ξ' is the geocentric tide of sea surface and δ is the geocentric displacement of the sea floor. With $\xi=\xi'-\delta$, the tide with respect to the sea floor, the total rate at which work is done is
$$\frac{\mathrm{d}W}{\mathrm{d}t}=\rho\psi\frac{\mathrm{d}\xi}{\mathrm{d}t}+\rho g(\xi+D)\frac{\mathrm{d}\delta}{\mathrm{d}t}+\rho\boldsymbol{\nabla}_s.[\boldsymbol{u}(\xi+D)\,\psi], \tag{35}$$
where ψ is the total tidal potential. Thus if the potential of the direct attraction by the tide raising potential is U_{lm}, (with $l=2$) and the potential of the ocean layer expanded according to (23*b*)
$$\begin{aligned}\psi &= (1+k_2)\,U_{lm}+\sum_s\alpha_s(1+k_s')\,\xi_s\\ &= (1+k_2)\,U_{lm}+\sum_s\sum_t\sum_{\pm}\alpha_s(1+k_s')\,D^{\pm}_{\beta,st}\cos\,[2\pi f_\beta T\pm t\lambda-\epsilon^{\pm}_{\beta,st}]\,P_{st}(\sin\varphi),\end{aligned} \tag{36}$$
where
$$\alpha_s = 4\pi GR\rho_w/(2s+1).$$
The total solid tide of the Earth will be
$$\delta=\frac{h_2}{g}U_{lm}+\sum_s\alpha_s\frac{h_s'}{g}\xi_s. \tag{37}$$
From (31*b*) and (35)
$$\langle\dot{W}\rangle=\rho_w\left\langle\psi\frac{\mathrm{d}\xi}{\mathrm{d}t}\right\rangle+\rho_w g\left\langle(\xi+D)\frac{\mathrm{d}\delta}{\mathrm{d}t}\right\rangle+\rho_w\langle\boldsymbol{\nabla}_s.\,[\boldsymbol{u}(\xi+D)\,\psi]\rangle.$$
For elastic yielding $\langle\mathrm{d}\delta/\mathrm{d}t\rangle=0$ and in the third term $\xi\ll D$. Hence
$$\langle\dot{W}\rangle=\rho_w\left\langle\psi\frac{\mathrm{d}\xi}{\mathrm{d}t}\right\rangle+\rho_w g\left\langle\xi\frac{\mathrm{d}\delta}{\mathrm{d}t}\right\rangle+\rho_w\langle\boldsymbol{\nabla}_s.\,[\boldsymbol{u}D\psi]\rangle,$$
and upon integration over the ocean surface the last term vanishes if the boundaries are impermeable. Then
$$\begin{aligned}\langle\overline{\dot{W}}\rangle &= \rho_w\int_S\left\langle\psi\frac{\mathrm{d}\xi}{\mathrm{d}t}\right\rangle\mathrm{d}S+\rho_w g\int_S\left\langle\xi\frac{\mathrm{d}\delta}{\mathrm{d}s}\right\rangle\mathrm{d}S\\ &= \rho_w I_1+\rho_w I_2.\end{aligned} \tag{38}$$
With (36) the first integral becomes
$$I_1=(1+k_2)\int_S\left\langle U_{lm}\frac{\mathrm{d}\xi}{\mathrm{d}t}\right\rangle\mathrm{d}S+\alpha_s(1+k_s')\int_S\left\langle\xi_s\frac{\mathrm{d}\xi_s}{\mathrm{d}t}\right\rangle\mathrm{d}S,$$
whose second part vanishes when integrated over the sphere. Similarly with the definition (37) of δ, the second integral of (38) reduces to
$$I_2=\frac{h_2}{g}\int_S\left\langle\xi\frac{\mathrm{d}U_{lm}}{\mathrm{d}t}\right\rangle\mathrm{d}S,$$
and
$$\langle\overline{\dot{W}}\rangle=\rho_w(1+k_2)\int_S\left\langle U_{lm}\frac{\mathrm{d}\xi}{\mathrm{d}t}\right\rangle\mathrm{d}S+\rho h_2\int_S\left\langle\xi\frac{\mathrm{d}U_{lm}}{\mathrm{d}t}\right\rangle\mathrm{d}S.$$

Thus apart from modifying the actual tide ξ, the Earth's deformation by the tidal load does not contribute to the mean rate at which work is done on the ocean layer and Hendershott's (1972) equivalent expression (52), restricted to the solid tide only, is valid in the more general case as well; on an elastically yielding Earth, the dissipation is still defined by the coefficient $D_{22}^{+} \sin \epsilon_{22}^{+}$. This means that $\langle \dot{W} \rangle$ can be estimated from empirical tide models without requiring any information on the manner in which the tide loads the Earth.

To reduce further these two integrals we write the potential as (from equation (1*a*) with (14*b*))

$$U_{lm} = U_{lmpq}^{0} P_{lm}(\sin \varphi) \begin{pmatrix} \cos \\ \sin \end{pmatrix}_{l-m \text{ odd}}^{l-m \text{ even}} (v_{lmpq} - m\lambda)$$

in which $v_{lmpq} \approx \pi(\frac{1}{2}r_\beta + m) - 2\pi f_\beta T$, and U_{lmpq}^{0} follows from (17) as

$$U_{lmpq}^{0} = \frac{Gm_{\text{☾}}}{a} \left(\frac{R}{a}\right)^{l} \frac{(l-m)!}{(l+m)!} (2 - \delta_{0m}) F_{lmp}(i) G_{lpq}(e). \tag{39}$$

Then with the tide defined by (23),

$$\int_S U_{lm} \frac{\mathrm{d}\xi}{\mathrm{d}t} \mathrm{d}S = -2\pi f_\beta U_{lm}^{0} D_{\beta, lm}^{+} N_{lm} \begin{pmatrix} \sin \\ \cos \end{pmatrix}_{l-m \text{ odd}}^{l-m \text{ even}} (2\pi f_\beta T - \epsilon_{\beta, lm}^{+} \pm v_{lmpq}),$$

and

$$\left\langle \int_S U_{lm} \frac{\mathrm{d}\xi}{\mathrm{d}t} \mathrm{d}S \right\rangle = 2\pi f_\beta U_{lm}^{0} D_{\beta, lm}^{+} N_{lm} \begin{pmatrix} \sin \\ \cos \end{pmatrix}_{l-m \text{ odd}}^{l-m \text{ even}} [\epsilon_{\beta, lm}^{+} \pi(\tfrac{1}{2}r_\beta + m)].$$

Similarly

$$\left\langle \int_S \xi \frac{\mathrm{d}U_{lm}}{\mathrm{d}t} \mathrm{d}S \right\rangle = 2\pi f_\beta U_{lm}^{0} D_{\beta, lm}^{+} N_{lm} \begin{pmatrix} \sin \\ \cos \end{pmatrix}_{l-m \text{ odd}}^{l-m \text{ even}} [\epsilon_{\beta, lm}^{+} \pi(\tfrac{1}{2}r_\beta + m)],$$

and

$$\langle \dot{W} \rangle = 2\pi f_\beta \rho_w (1 + k_2 - h_2) U_{lm}^{0} D_{\beta, lm}^{+} N_{lm} \begin{pmatrix} \sin \\ \cos \end{pmatrix}_{l-m \text{ odd}}^{l-m \text{ even}} [\epsilon_{\beta, lm}^{+} \pi(\tfrac{1}{2}r_\beta + m)].$$

In these expressions

$$N_{lm} = \frac{4\pi R^2 (l+m)!}{(2l+1)(l-m)!(2-\delta_{0m})}.$$

With (39),

$$\langle \dot{W} \rangle = 2\pi f_\beta (1 + k_2 - h_2) \frac{4\pi 6 R^2 m_{\text{☾}} \rho_w}{a} \left(\frac{R}{a}\right)^{l} \frac{D_{\beta, lm}^{+}}{2l+1} F_{lmp}(i) G_{lpq}(e) \begin{pmatrix} \sin \\ \cos \end{pmatrix}_{l-m \text{ odd}}^{l-m \text{ even}} [\epsilon_{\beta, lm}^{+} \pi(\tfrac{1}{2}r_\beta + m)]. \tag{40}$$

The work method, integral (32), therefore requires only the second degree harmonics in the ocean tide. More specifically, only the harmonics with the same degree and order as the potential of the forcing function intervene in the mean rate at which energy is dissipated in the global oceans; the rate at which work is done by the other harmonics being zero when averaged over one period and over the world's oceans. This simple result has apparently been overlooked in the literature on tidal dissipation. Its consequence is that the rate of energy dissipation can be computed with relatively good precision since the second degree terms of the various M_2 models are in quite good agreement despite the variance in their detail.

Table 5 summarizes several estimates of $\mathrm{d}E/\mathrm{d}t$ for the M_2 tide based on equations (40) and on the coefficients $D_{22}^{+} \sin \epsilon_{22}^{+}$ given in table 4. The mean value for the rate of dissipation of the M_2 tide is $(3.3 \pm 0.3) 10^{19}$ erg s^{-1}. For comparison, $\mathrm{d}E/\mathrm{d}t$ computed from some of these models by other authors are also given. Hendershott (1972) integrates the work done on the ocean by the Moon as well as by the elastic body tide. The latter reduces the dissipation rate from what it would be for the same tide on a rigid Earth by a factor $(1 + k_2 - h_2)$ or by about 70% (equation 40). The result by Pariyskiy *et al.* (1972) based on the model by Bogdanov & Magarik (1967), and by

Kuznetsov (1972) based on the models by Pekeris & Accad (1969), and by Zahel (1970) must likewise be reduced by this factor. Finally, as the Pekeris & Accad and Zahel (1970) models correspond to tides on a solid Earth, an additional reduction of 70% is in order to allow for the modification of the ocean tide by the elastic tide. The agreement between these results and those obtained from equation (40) is now in general within 10%.

The result (40) for the rate at which work is done on the ocean surface can also be estimated directly from the energy balance in the Earth–Moon system. From equation (3*a*) with the tidal acceleration given by (2*b*) and the secular rates in the orbital elements given by (27), ignoring terms in e^2 and with $Mm_☾/(M+m_☾) \approx m_☾$, this method gives

$$\frac{\mathrm{d}E}{\mathrm{d}t} = [n(l-2p+q)+m\dot{\theta}]\,\frac{4\pi GR^2 m_☾ \rho_w}{a}\,\frac{(1+k_l')}{2l+1}\left(\frac{R}{a}\right)^l \times F_{lmp}(i)\,G_{lpq}(e)\,D^{+}_{\beta,lm}\begin{bmatrix}\sin\\ \cos\end{bmatrix}^{l-m\text{ even}}_{l-m\text{ odd}} \epsilon^{+}_{\beta,lm}. \tag{41}$$

The frequency of the tidal wave is

$$-2\pi f_\beta \approx (l-2p)\,\dot{\omega}+(l-2p+q)\,n+m(\dot{\theta}-\dot{\Omega}),$$

but as $\dot{\omega}$ and $\dot{\Omega}$ are both small compared with n or $\dot{\theta}$,

$$-2\pi f_\beta \approx (l-2p+q)\,n+m\dot{\theta}.$$

The only difference between the expressions (40) and (41) now is the factor $(1+k_2-h_2)$ appearing in the former and $(1+k_2')$ in the latter, but the two are numerically equal.†

Table 6. Summary of ocean tide parameters: χ_{lmpq} is the factor from equation (28) for estimating the equivalent solid Earth lag angles ϵ_{lmpq}

	D^{+}_{22} or D^{+}_{21} cm	ϵ^{+}_{22} or ϵ^{+}_{21}	$D^{+}_{22}\sin\epsilon^{+}_{22}$ or $D^{+}_{21}\cos\epsilon^{+}_{21}$	χ_{lmpq} cm^{-1}	ϵ_{lmpq}
ocean model results					
M_2	3.64	112°	3.37 ± 0.30	0.033	6.4
S_2	1.66	134	1.19 ± 0.25	0.071	4.4(3)
N_2	0.69	100	0.68 ± 0.20	0.170	6.6
K_2 (Lunar)	0.32	136	0.22 ± 0.07	0.364	4.6
K_2 (Solar)	0.14	136	0.10 ± 0.03	0.793	4.6
L_2	0.10	−56	-0.08 ± 0.02	−1.189	5.5
$2N_2$	0.09	88	0.09 ± 0.03	1.274	6.6
T_2	0.10	133	0.07 ± 0.02	1.187	4.8
K_1 (Lunar)	1.37	236	-0.77 ± 0.15	−0.041	1.8
K_1 (Solar)	0.63	236	-0.35 ± 0.07	−0.087	1.8
O_1	1.17	50	0.75 ± 0.15	0.039	1.7
P_1	0.54	55	0.31 ± 0.10	0.084	1.5
Q_1	0.22	47	0.15 ± 0.05	0.201	1.7
atmospheric tides					
S(1)			-0.10 ± 0.05		
S(2)			-0.32 ± 0.10		
satellite results					
M_2			2.6 ± 0.6		4.9
S_2			1.3 ± 0.2		5.1(3)

(1) To be used with the ocean model results.
(2) To be used with the satellite results.
(3) Equivalent phase angle for combined ocean and atmosphere.

† Recently S. M. Molodensky has demonstrated the equivalence of these two factors.

8. Comparisons and discussion

Summary of ocean tide parameters

Table 6 summarizes the ocean tide parameters required for estimating the tidal accelerations.

M_2: The adopted values represent the mean of the four independent models 1–4 of table 4. The amplitudes have been corrected for the tidal yielding when appropriate. The individual values differ from the mean by less than 10% and an uncertainty of this amount is adopted for the $D_{22}^{+} \sin \epsilon_{22}^{+}$.

S_2: The age of the semi-diurnal tide is taken as 0.9 days, the mean of the results obtained by Garrett & Munk (1971) and from the Bogdanov & Magarik models. The age added to the phase of the M_2 tide estimated above, determines the phase of the S_2 tide. The ratio $D_{M_2, 22}^{+}/D_{S_2, 22}^{+}$ from the Bogdanov & Magarik models is 2.3, close to the equilibrium value of 2.1. For $D_{S_2, 22}^{+}$ the mean value for the M_2 tide coefficient is scaled by the mean of these two ratios. An uncertainty of 20% is assumed.

Atmospheric S_2: Both the age of the tide and the S_2 model will be contaminated by the atmospheric loading and a residual atmospheric tide over the continents of

$$(1 - a_{00}) \, (D_{22}^{+} \sin \epsilon_{22}^{+})_{\text{total atmosphere}}$$

must be taken into account. This contribution is small compared with the uncertainties in the principal ocean tide coefficients.

Other semi-diurnal tides: Amplitudes and lags of the other tides are computed according to equations (31), using the mean M_2 amplitude and a phase lag between S_2 and M_2 of 22° corresponding to the above mean age of the semi-diurnal tide. Uncertainties for these tidal parameters are assumed to be 30%.

Diurnal tides: The means of the Dietrich and Bogdanov & Magarik solutions are adopted for both the K_1 and O_1 tides with uncertainties of 20%. The amplitude ratio of the adopted values is 1.7 compared with 1.4 for the equilibrium ratio. The phase difference is close to the value estimated directly from Dietrich's observed diurnal ages. The lunar and solar parts of K_1 are separated according to their equilibrium ratio. For the other diurnal tides of marginal importance we use equations (31) after substituting K_1 and O_1 for S_2 and M_2. Uncertainties of 30% are assumed. In Lambeck's (1975) analysis the K_1 and O_1 values were estimated from Dietrich's solution with extrapolated amplitudes that lead to amplitudes for D_{21}^{+} that are considerably larger than the equilibrium values.

Satellite results: For convenience the satellite results for the M_2 and S_2 tides are repeated in table 6. The latter includes the combined effect of the ocean and atmospheric tides. The M_2 tide solution is accurate to 20%, S_2 to about 10%.

Lunar acceleration

From the equations (27) and the ocean tide parameters summarized in table 6 the secular changes in the Moon's orbital elements ($\mathrm{d}a/\mathrm{d}t$, $\mathrm{d}e/\mathrm{d}t$, $\mathrm{d}i/\mathrm{d}t$ and $\dot{n}$) can be evaluated (table 7). Of these elements, the latter can be directly compared with the astronomical estimate for the Moon's acceleration in longitude. The principal contribution to $\dot{n}$ comes from the M_2 tide with smaller contributions coming from N_2 and O_1. The use of all relevant tidal frequencies gives a total acceleration in longitude of $(-30.6 \pm 3.1)'' \mathrm{cy}^{-2}$. The satellite estimate of the M_2 tide parameter tends to be smaller than the ocean model result and if we scale the other contributions by a similar ratio the satellite based estimate for the lunar acceleration is $(-27.3 \pm 5.2)'' \mathrm{cy}^{-2}$. Both values are consistent with Muller's (1975) 'best estimate' of $-28'' \mathrm{cy}^{-2}$ based on several astronomical sources (§3). The good agreement between the latter and the satellite result is better than we have the right to expect in view of their rather large error estimates. But this agreement does indicate that we have a powerful new method of estimating the tidal accelerations and improved results can be expected when long series of observations of satellites such as

TABLE 7. ESTIMATES OF SECULAR TIDAL CHANGES IN a e i DUE TO THE OCEAN TIDES COMPARED WITH ASTRONOMICAL AND SATELLITE ESTIMATES

tide	$\mathrm{d}a/\mathrm{d}t$ / $10^{-7}(\mathrm{cm\ s^{-1}})$	$\dot{n}$ / $10^{-23}\ \mathrm{s^{-2}}$	$\mathrm{d}e/\mathrm{d}t$ / $10^{-19}\ \mathrm{s^{-1}}$	$\mathrm{d}t/\mathrm{d}i$ / $10^{-19}\ \mathrm{s^{-1}}$	% error estimate	satellite solution / $\dot{n}/(10^{-23}\ \mathrm{s^{-2}})$
M_2	1.29	−1.34	−0.45	−3.46	10	−1.03
N_2	0.08	−0.08	5.82	−0.14	30	−0.07
K_2	—	—	—	−0.02	30	—
L_2	—	—	−0.10	—	30	—
$2N_2$	—	—	0.21	—	30	—
K_1	—	—	—	−1.38	20	—
O_1	0.07	−0.07	−0.02	0.80	20	−0.07
O_1	—	—	0.27	0.03	30	—
total	1.44	−1.49	5.73	−4.17	—	−1.17
	±0.15	±0.15	±1.75	±0.47	—	±0.25

GEOS 3 and STARLETTE become available. While the agreement between the astronomical and oceanic estimates of $\dot{n}$ is such that the principal role of the oceans in dissipating the tidal energy is established beyond any doubt, the uncertainties of both estimates are still uncomfortably large: in particular we cannot draw any firm conclusions about the possible role of dissipation in the solid parts of the Earth and Moon.

Eccentricity, inclination and lunar node

The present tidal variations in the eccentricity and inclination of the lunar orbit follow from (27) and the rates are small (table 7). The former is of the order $5 \times 10^{-19}\,\mathrm{s^{-1}}$, very much smaller than the value of $(1.5 \pm 0.6)\ 10^{-16}\,\mathrm{s^{-1}}$ deduced by Martin & Van Flandern (1970) from the lunar observations. Tides raised on the Moon are also quite inadequate to explain this difference and the explanation for the observed value must be sought elsewhere; it cannot be caused by tidal dissipation as these authors suggest. If Martin & Van Flandern's results are confirmed this would suggest remaining long-period discrepancies in the lunar and solar theories which may also explain the different values for $\dot{n}$ based on the telescope observations since the seventeenth century, modern observations with respect to the atomic time scale, and the eclipse solutions. The present tidal change of the inclination of the lunar orbit on the equatorial plane is insignificantly small as will be the inclination of the equator on the ecliptic (equation 67 of Kaula 1964). Martin & Van Flandern's analysis of the lunar motion does not indicate a significant variation in these elements.

In most discussions of the tidal dissipation problem it is assumed that the tidal potential does not introduce a secular rate in the lunar node due to a zero tidal torque about the line of nodes (see for example, Jeffreys 1962; MacDonald 1964), but as stressed by Kaula (1964), this is valid only if all phase lags ϵ_{lmpq} in the potential (19) are equal. From the Lagrangian planetary equations for Ω, ignoring indirect effects due to the Earth's oblateness and solar attraction,

$$\dot{\Omega}_{2mpq} = \frac{k_2}{na^2(1-e^2)^{\frac{1}{2}}\sin i}\left(\frac{R}{a}\right)^5 \frac{Gm_{☾}}{a}(2-\delta_{0m})\frac{(l-m)!}{(l+m)!}F_{lmp}(i)\,[G_{lpq}(e)]^2\frac{\partial F_{lmp}(i)}{\partial i}\cos\epsilon_{2mpq}, \quad (42)$$

and for constant ϵ_{2mpq} the sum

$$\sum_m\sum_p\sum_q (2-\delta_{0m})\frac{(l-m)!}{(l+m)!}F_{lmp}(i)\frac{\partial F_{lmp}(i)}{\partial i}[G_{lpq}(e)]^2$$

vanishes. The assumption of constant phase lags is not evident, however, if the dissipation occurs in the oceans, and the equivalent phase lags (table 6) can be expected to vary with tidal period; in particular, long period tides may follow the equilibrium tide more closely than the semidiurnal tides. A more reasonable assumption may be to assume that the phase lags of all tides of the same fundamental frequency are constant. Then (42) gives

$$\dot{\Omega} = 2.29\times10^{-18}\cos i\{-(1+3\cos^2 i)\cos\epsilon_{22}+4(3\cos^2 i-1)\cos\epsilon_{21}+(5-9\cos^2 i)\cos\epsilon_{20}\}, \quad (43)$$

where ϵ_{22} now refers to the equivalent phase lag of the semi-diurnal tides, ϵ_{21}, that of the diurnal tides and ϵ_{20} that of the zonal tides. The principal contributions to the semi-diurnal tide is M_2, to the diurnal tide O_1 and K_1 (both the lunar and solar parts) while the principal perturbation to the long period tide comes from $lmpq = 2010$ which is a secular tide when referred to an Earth fixed reference frame and presumably has zero phase lag. However, all lag angles are small and $\cos\epsilon_{22}\approx\cos\epsilon_{21}\approx\cos\epsilon_{20} = 1$ and $\dot{\Omega}$ will be small; much smaller than the value of $(4.3\pm0.4)''\,\mathrm{cy}^{-1}$ observed by Martin & Van Flandern (1970), and 4.4″ computed by Muller, Newhall, Van Flandern and Williams (see Muller 1976).

Owing to a combined Earth, Sun and planetary effect, the lunar node varies linearly with time, at a rate $\dot{\Omega}'$ that is a function of a, e and i. However, because of the tidal changes, the latter elements will also vary with time and the lunar node will experience an acceleration according to

$$\ddot{\Omega} = \frac{\partial}{\partial a}(\dot{\Omega}')\,\dot{a}+\frac{\partial}{\partial i}(\dot{\Omega}')\frac{\mathrm{d}i}{\mathrm{d}t}+\frac{\partial}{\partial e}(\dot{\Omega}')\frac{\mathrm{d}e}{\mathrm{d}t},$$

in which the most important contribution comes from the first term. According to Muller (1975)

$$\ddot{\Omega} \approx \frac{\partial}{\partial a}(\dot{\Omega}')\,\dot{a} = \frac{\partial}{\partial n}(\dot{\Omega}')\,\dot{n} \approx 0.0037\dot{n}'\,\mathrm{cy}^{-2} \quad (44)$$

and with the observed value of $\dot{n}\approx -28''\,\mathrm{cy}^{-2}$, $\ddot{\Omega}\approx -0.11''\,\mathrm{cy}^{-2}$.

The tides will also introduce linear and quadratic rates in the argument of perigee and in the mean anomaly of the Moon which can be estimated in the same manner as the above rates in the ascending node, and they will be of the same magnitude. These effects will introduce small errors in the astronomical estimates of the lunar acceleration since in the lunar orbital theory one writes for the mean lunar longitude

$$\lambda = A_o + B_t T + \tfrac{1}{2}(C_o + C_t)\,T^2,$$

where the subscripts o denote observed values and the subscripts t theoretical values based on a gravitational theory. The tidal acceleration then follows from $\dot{n} = C_o$. In a precise analysis one should write, instead of B_t, $(B_t+\dot{\Omega}+\dot{\omega}+\dot{M})$ and for C_t, $(C_t+\ddot{\Omega}+\ddot{\omega}+\ddot{M})$ where $\dot{\Omega}$, $\dot{\omega}$, $\dot{M}$ represent the linear rates in the Moon's elements due to the effect of the tides and $\ddot{\Omega}$, $\ddot{\omega}$, $\ddot{M}$ represent the quadratic indirect effects of the type (44). The present level of accuracy does not appear to warrant this refinement.

The Earth's secular acceleration

The tidal acceleration of the Earth follows from (2*b*) where da/dt, de/dt and di/dt follow from (27). Both lunar and solar tides must be considered. For the latter the changes in the Earth's orbit are negligible but the effect of the solar torque on the spin is not, due to the $a_{\odot}^2$ term entering into the solar equivalent of equation (2*b*). The total tidal acceleration is

$$\ddot{\theta}_{\mathrm{T}} = (\ddot{\theta}_{\mathrm{T}}|_a + \ddot{\theta}_{\mathrm{T}}|_e + \ddot{\theta}_{\mathrm{T}}|_i)_{☾} + (\ddot{\theta}_{\mathrm{T}}|_a + \ddot{\theta}_{\mathrm{T}}|_e + \ddot{\theta}_{\mathrm{T}}|_i), \tag{45}$$

TABLE 8. ESTIMATES OF THE EARTH'S SECULAR ACCELERATION $\ddot{\theta}_{\mathrm{T}}$ FROM OCEAN MODELS, ASTRONOMICAL AND SATELLITE OBSERVATIONS. $\ddot{\theta}_{\mathrm{T}}|_{\kappa_i}$ IS THE ACCELERATION DUE TO A CHANGE IN THE LUNAR ORBIT ELEMENT κ_i

tide	$\ddot{\theta}_{/a}$	$\ddot{\theta}_{/e}$ $10^{-22}\,\mathrm{s}^{-2}$	$\ddot{\theta}_{/i}$	total $\ddot{\theta}_{\mathrm{T}}$ from satellite solutions
M_2	−5.44	−0.01	−0.49	−4.70
N_2	−0.33	0.10	−0.02	(−0.25)
K_2	—	< 0.01	< 0.01	—
$2N_2$	—	< 0.01	—	—
S_2	−0.90	—	−0.08	−1.25
T_2	−0.06	0.02	—	(−0.04)
K_1	—	—	−0.20	(−0.20)
O_1	−0.28	—	0.11	(−0.17)
Q_1	—	< 0.01	—	—
P_1	−0.05	—	0.02	(−0.03)
S_2 (atmos)	0.08	—	0.01	0.24
total	−6.98	0.11	−0.65	−6.40 ± 1.50
		−7.52 ± 0.75 (sum of the three columns)		

where $\ddot{\theta}_{\mathrm{T}}|_{\kappa_i}$ denotes the contribution of the total acceleration due to the secular change in the element κ_i. Of the various contributions to (45) (table 8) the dominant part, some 80 %, comes from $(\ddot{\theta}_{\mathrm{T}}|_a)_{☾}$ and this quantity can be estimated either from the tidal theory or deduced from the astronomically observed $\dot{n}$. From the former, table 7, $(\ddot{\theta}_{\mathrm{T}}|_a)_{☾} = -6.05 \times 10^{-22}\,\mathrm{s}^{-2}$ while the astronomical data gives $-5.53 \times 10^{-22}\,\mathrm{s}^{-2}$. The total oceanic estimate of the $\ddot{\theta}_{\mathrm{T}}$ is $(-7.7 \pm 0.8) \times 10^{-22}\,\mathrm{s}^{-2}$. The satellite solution (for tides other than M_2 and S_2 the ocean models have been used) gives $(-7.3 \pm 1.5) \times 10^{-22}\,\mathrm{s}^{-1}$ and the astronomical estimate [$(\ddot{\theta}_{\mathrm{T}}|_a)_{☾}$ from the observed $\dot{n}$ in addition to ocean estimates for $\ddot{\theta}_{\mathrm{T}}|_e$, $\ddot{\theta}_{\mathrm{T}}|_i$ and solar tides] gives $(-7.2 \pm 0.7) \times 10^{-22}\,\mathrm{s}^{-2}$. With these values, the total tidal acceleration of the Earth is given as function of the Moon's tidal acceleration in longitude by

$$\ddot{\theta}_{\mathrm{T}} = 53.1\dot{n}. \tag{46}$$

The astronomical evidence for the observed acceleration of the Earth has been reviewed in §3. The non-tidal acceleration of the Earth $\ddot{\theta}_{\mathrm{NT}} = \ddot{\theta} - \ddot{\theta}_{\mathrm{T}}$ is the most unsatisfactory quantity due to it being the difference between two quantities, both of limited accuracy. If the mean of the above three results for $\ddot{\theta}_{\mathrm{T}}$ is adopted, then $\ddot{\theta}_{\mathrm{NT}} \approx 2.0 \times 10^{-22}\,\mathrm{s}^{-1}$. With both $\ddot{\theta}$ and $\ddot{\theta}_{\mathrm{T}}$ precise to about 10 %, the precision of $\ddot{\theta}_{\mathrm{NT}}$ is about 50 % of its absolute value and its significance becomes difficult to ascertain. This value for $\ddot{\theta}_{\mathrm{NT}}$ is less than that estimated by Lambeck (1975) due to (i) Muller's (1975, 1976) revision of the astronomical data and (ii) the present revision of the tide data. Muller (1976) finds a further reduction in $\ddot{\theta}_{\mathrm{NT}}$ when he solves for a change in gravitational constant in addition to the tidal accelerations. If we introduce into his solution the revised tidal estimates, $\ddot{\theta}_{\mathrm{NT}}$ is further reduced and becomes insignificant. We are rapidly approaching the embarrassing situation of a phenomenon for which there has never been a shortage of geophysical explanations now appearing to be vanishing. It is reminiscent of the earlier discussion of tidal dissipation and stresses, once again, the need for further improvements in both the observed and theoretical accelerations. Now we cannot seek comfort in new methods such as lunar laser ranging or satellite orbit analyses since $\ddot{\theta}$ is the sum of the secular part and long period irregularities. Only ancient astronomical observations can contribute and this emphasizes the need for a systematic search for new records going further back into time than the presently available data.

Energy dissipation

The amount of tidal energy that must be dissipated in the Earth–Moon system is given by equation (3) as a function of the lunar orbital acceleration and of the Earth's rotational acceleration. That is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = C\dot{\theta}\ddot{\theta}_{\mathrm{T}} - \tfrac{1}{3}m_{☾}\, n a_{☾}^{2}\dot{n}_{☾},$$

where $\ddot{\theta}_{\mathrm{T}}$ is the total tidal acceleration of the Earth. With (46)

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 3.02\times 10^{42}\dot{n}_{☾}\ \mathrm{erg\,s^{-1}}$$

or, with the acceleration given in § 3, the astronomical estimate is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 4.11\times 10^{19}\ \mathrm{erg\,s^{-1}}.$$

The satellite solution gives a comparable $3.6\times 10^{-19}\ \mathrm{erg\,s^{-1}}$ and the difference between this and the astronomical estimate leads to an estimate of the amount of energy dissipated in the Moon. Present results are inadequate apart from confirming that dissipation in the Moon is small.

The total rate of energy dissipation in the oceans is a comparable $4.5\times 10^{19}\ \mathrm{erg\,s^{-1}}$, stressing once again that a very major part of the tidal energy is dissipated in the oceans and that the solid Earth does not possess an important energy sink. Energy dissipated in the M_2 tide is 3.06×10^{19} $\mathrm{erg\,s^{-1}}$ (astronomical estimate) or $3.35\times 10^{19}\ \mathrm{erg\,s^{-1}}$ (tidal model estimate).

Limits on mantle Q

In view of the uncertainties in the two estimates of $\mathrm{d}E/\mathrm{d}t$, the fact that the tidal estimate is somewhat greater than the astronomical estimate is not significant, in particular since the satellite results suggest that the tidal estimates may be too high. If we take the difference between the upper limit, $-31\ \mathrm{cy^{-1}}$, of the astronomical estimate for $\dot{n}$ and the lower limit, $27.6''\ \mathrm{cy^{-1}}$ estimated from the tide models, we obtain what can be considered as an estimate of the maximum specific dissipation of the Earth. From (21),

$$\delta\dot{n} = -\frac{3n}{2a}\sum_{lmpq} 2\kappa_{lm}[F_{lmp}(i)\,G_{lpq}(e)]^{2}\,(l-2p+q)\sin\epsilon_{lmpq}, \tag{47}$$

where the contribution of Moon tides to $\delta\dot{n}$ is neglected. The three principal contributions to (47) come from the M_2, N_2 and O_1 tides and we assume that the phase lags ϵ_{lmpq} are constant for these three frequencies. Writing

$$\sin\epsilon_{lmpq} \approx \tan\epsilon_{lmpq} = Q^{-1},$$

we obtain $Q^{-1} < 90^{-1}$. More precise upper limits for the specific dissipation can only be established if both the astronomical data and the tide models are improved.

The mantle Q can also be estimated, in principal at least, from a comparison of the satellite and numerical results for the coefficients $D_{22}^{+} \sin \epsilon_{22}^{+}$ (or the $D_{21}^{+} \cos \epsilon_{21}^{+}$) since the former is a measure of the total response of the Earth to the tidal potential. From the results for M_2 summarized in table (6), the difference between the upper limit of the satellite solution and the lower limit of the model solution results in an equivalent residual phase lag of 0.2°, yielding a solid Earth Q of 300 or more. For the S_2 solution the satellite coefficients are somewhat larger than the ocean model coefficients and the difference could be interpreted as a measure of dissipation in the mantle. From table 6 the result is $Q \approx 250$ with limits between 160 and 480. The mean of the above three estimates for Q^{-1} leads to a lower limit for the mantle Q at the tidal frequencies of about 120. An improved value for Q requires (i) better satellite results, (ii) an improved ocean model and (iii) correct treatment of the ocean–atmosphere interaction. The advantage of this approach is that the Q at diurnal frequencies can also be established, once more satellite orbits have been analysed and once more reliable models for the diurnal tides become available.

Ocean dissipation mechanisms

The bottom friction and energy flux calculations by Jeffreys, Heiskanen and Miller all suggest that the dissipation is very localized. Thus in Miller's (1966) calculation, about 14 % of the total energy is dissipated in the Bering Sea and 12 % in the Okhotz Sea. The Timor Sea, Patagonia shelf and Hudson Strait account for another 24 % and ten smaller seas contribute a further 30 %. In the earlier studies the energy sink in the Bering Sea was even more important, 70 % in the case of Jeffreys' (1920) study and 25 % in the case of Heiskanen's study (1921). If these estimates are of the correct magnitude then the phase of the global ocean tide must be very significantly modified by the dissipation in these shallow seas.

Dissipation in the Bering Sea has dominated all earlier discussions. However, the tidal currents across the shelf seem to be less important than the values used by Jeffreys, Heiskanen and Miller. Maximum tidal currents around the Pribilof and St Mathew islands on the edge of the Bering shelf and elsewhere have an average value of less than 2 km/h and the open sea currents are likely to be less than 1 km/h. Tidal amplitudes are of the order of 20 cm (National Ocean Survey, 1975). The average depth of the shelf margin is about 60 m and its length is about 1500 km. The energy flux method (integral (34)) then gives

$$-\mathrm{d}E/\mathrm{d}t \approx 5 \times 10^{17}\ \mathrm{erg\ s^{-1}}.$$

This is a maximum value and includes tides other than M_2. It is nearly an order of magnitude less than the value found by Miller (1966). For $\alpha = 0.002$ and a shelf area of $1.1 \times 10^{6}\ \mathrm{km^2}$ the bottom friction method (integral (332)) gives the same value for $\mathrm{d}E/\mathrm{d}t$. It appears most unlikely that the Bering Sea can play the dominant role that is suggested by the calculations of Jeffreys, Heiskanen and Miller and, if from the above results we can extrapolate to other seas, Miller's total of $1.7 \times 10^{19}\ \mathrm{erg\ s^{-1}}$ represents very much an upper limit to the amount of energy that can be dissipated by bottom friction in shallow seas. Admittedly his estimates are based on very few data indeed but if the history of this calculation has shown one thing it is that the estimates decrease with increasing information.

A hint that the bottom friction calculations may not be in order is already given by Hendershott (1973). The lower limit to the Q of the global ocean, as estimated by Garrett & Munk (1971), is of the order 25. Hendershott (1972) estimates a Q of 34. But the analysis by Wunsch (1972) of the North Atlantic tide suggests a lower limit to Q of about 5, much smaller than the global estimate, and unexpected from Miller's (1966) calculations which indicate that the North Atlantic is relatively dissipationless.

If dissipation does occur in a few local areas by severe bottom friction or by some other mechanism, the second degree terms in the tide must be significantly modified by the tides in these areas. From the various model calculations this does not appear to be so; the agreement between the estimates of $\mathrm{d}E/\mathrm{d}t$ from tide models based on quite different assumptions about the way dissipation is introduced, suggests that more global factors control the second degree harmonics. Both Hendershott and Bogdanov & Magarik exclude some of the important shallow seas from their solutions; Pekeris & Accad define the continent–ocean margin by the 1000 m depth contour and assume linear friction; Zahel allows for dissipation by turbulence, yet all yield quite similar results for $\mathrm{d}E/\mathrm{d}t$. This is perhaps as it should be. The torques exerted on the Earth by the Moon are described by the second degree harmonics but the energy is dissipated by components at the other end of the wavelength spectrum. To estimate the torques we are only interested in these second degree terms, particularly in the phase lag, and what happens to the energy once it passes into higher modes need concern us no further. The efficacity of this breakup into the higher harmonics is presumably dominated by global ocean characteristics since the above calculations yield essentially the same results for $\mathrm{d}E/\mathrm{d}t$ despite the differences in methods. Geometry of the ocean–continent configuration, continental margins and sea floor topography would appear to be more important than what happens to the high frequency part of the spatial spectrum in, say, a few localized shallow seas. To evaluate the lunar acceleration of the Moon by estimating dissipation in shallow seas, one is in fact trying to re-establish the second degree harmonic of the tide from very localized measurements and clearly this is a difficult and uncertain exercise at best.

Dissipation over the coastal shelves may be more important than is generally supposed.† Defant (1961) suggests that the average tidal currents over these shelves is of the order of 0.5 knots or $30\,\mathrm{cm\,s^{-1}}$, leading to a dissipation rate of about $50\,\mathrm{erg\,s^{-1}\,cm^{-2}}$. The total shelf area is of the order of $30\times10^{6}\,\mathrm{km^{2}}$ resulting in a total rate of dissipation of the order of $1.5\times10^{19}\,\mathrm{erg\,s^{-1}}$, nearly one half of the astronomically required value for the M_2 tide. Munk (1968) suggests that, due to an interaction with internal tides, the tidal currents at the bottom of the deep oceans may be larger than generally thought so that the deep sea may yet be an important energy sink. His provisional estimate is $10^{18}\,\mathrm{erg\,s^{-1}}$. Jeffreys (1929, 1962) has suggested that dissipation along the open coast lines may be important since ordinary waves breaking on the coast are almost totally dissipated, there being a general absence of strong reflected waves along the shore. Jeffreys (1968) discusses this possibility in some detail and concludes that dissipation by the breaking of the waves is more important than by bottom friction. Applying the former mechanism to tidal waves, he concludes that it may be an important source for the loss of tidal energy. Jeffreys proposes a boundary condition to take into account this mechanism,

$$u = (g/D)^{\frac{1}{2}}\xi$$

between the velocity u, water depth D and tide amplitude ξ at a point a suitable distance from the shore (see also Proudman 1941). This leads to a dissipation (with equation (34*b*))

$$-\left\langle\frac{\mathrm{d}E}{\mathrm{d}t}\right\rangle = g^{\frac{3}{2}}\rho\int_L\int_T D^{\frac{1}{2}}\xi^2\,\mathrm{d}T\,\mathrm{d}L.$$

Such a boundary condition has not yet been applied in global tide models.

† See also Webb (1976).

Other proposed mechanisms for tidal dissipation are not much better at quantifying the actual rate at which energy is dissipated. Munk (1968) concludes that a significant fraction of the dissipation may take place by way of scattering into internal modes, either by the sea floor topography, along coastlines or along the outer edge of the continental margins. Once the energy is in the internal modes it can then be dissipated in a number of ways, for example in the shear layer above the deep ocean floor or along surfaces of discontinuity in density. Cox & Sandstrom (1962) suggest that a significant amount of tidal energy may be scattered into internal modes by an irregular bottom topography and, according to Munk (1968) their theory yields an energy flux of 5×10^{18} erg s^{-1}. Wunsch & Hendry (1972) have measured the rate of such a conversion on the continental slope of New England and, if their value is representative of the world wide continental slopes, the total rate of energy conversion is only 10^{16} erg s^{-1} (Wunsch 1975). Garrett & Munk (1972) estimated the dissipation through the breaking of internal waves and conclude that this may amount to about 7×10^{18} erg s^{-1}. According to Wunsch (1975) this value must be considered as an upper limit. Finally, Leblond (1966) has investigated dissipation of internal waves by turbulent friction and finds that such a mechanism may be important for the tidal waves. But as stressed by Wunsch, the estimate of the total rate of dissipation is very dependent on the value for the vertical eddy viscosity coefficient and the value of 10^2 cm^2 s^{-1} adopted by Leblond is higher than usually assumed.

Whichever of these mechanisms is responsible for the dissipation of energy, there is evidence for the oceans to be close to resonance at the semi-diurnal frequency. If the phase lags of the M_2 and S_2 frequencies were the same then the age of the tide would be zero instead of the observed 1 day. Satellite results also indicate a different lag ϵ_{22}^{+} and hence a different Q for these two tides. Numerical models of the M_2 tide also show some sensitivity to small changes in the model and Pekeris & Accad (1969) suspect that this tide is close to a resonance frequency. Calculations by Longuet-Higgins & Pond (1970) and in particular by Platzman (1975) show that the oceans may possess several free modes whose frequencies are close to semi-diurnal. Platzman also finds a free mode with a frequency near diurnal. This suggests strongly that changes in the ocean continent geometry can have had important consequences on the dissipation in the past. In particular if in the past, there existed ocean geometries that result in free modes with frequencies distinctly different from the forcing frequency, the rate of dissipation could have been significantly less than its present value. As both the frequencies of the free modes and of the forcing function will vary slowly with time due to the secular tidal acceleration of the Earth, even if all other factors have remained constant, it does not appear feasible to determine if the present near-resonance conditions have existed or not over long time intervals during the past, without solving the free oscillation problem for each case.

Constancy of tidal dissipation

Newton (1970) suggested that the lunar acceleration may have undergone important changes over the last 3000 years. He concluded this from his results based on satellite observations of the tidal perturbations and on his analysis of ancient and medieval eclipse records. His satellite results, corresponding to a present day value for the dissipation, for $\dot{n}_{☾}$ are close to the Spencer-Jones determination of $22''/\mathrm{cy}^2$ but this agreement must be considered as fortuitous rather than real and his results (Newton 1968) must be discarded for the following reasons:

(1) Newton does not allow for the fourth degree harmonics in the ocean tide; for the satellites used these terms are as important as the second degree harmonics.

(2) He does not allow for the frequency dependence of the tide coefficients. This is particularly important as the M_2 and O_1 tidal perturbations cannot be separated from his data.

(3) His treatment of the loading of the Earth by the atmospheric tide is incorrect.

(4) The dispersion of individual results for the lunar and solar tides obtained from the perturbations in inclination and ascending node of four satellites is far greater than can be explained by the above effects and is indicative of further unmodelled perturbations in the orbital theory.

Newton's values for $\dot{n}_{☾}$ at epochs 200 B.C. and 1000 A.D. $(41.6 \pm 4)''/\text{cy}^2$ and $(-42.3 \pm 6)''/\text{cy}^2$ respectively, are not significantly different. Newton's (1972) value of $(-79 \pm 16)''/\text{cy}^2$ centred at epoch 1000 A.D. is quite different but Muller (1975) argues that this value is in error due to Newton's use of partial eclipse records. Muller & Stephenson (1975) and Muller (1975) find no evidence for a change in the lunar acceleration. This makes good geophysical sense since a variation in $\dot{n}_{☾}$ by a factor of about 2 as suggested by Newton (1970) requires a comparable change in the coefficients $D_{22}^{+} \sin \epsilon_{22}^{+}$. Newton (1970) suggests that important dissipation may occur by friction between the ocean and shelf ice, implying that the shelf ice controls the tidal bulge. Whatever the merits of this mechanism, there is no evidence that significant changes have occurred in the extent of the shelf ice since sea level has not changed by more than a few metres during the last three thousand years. Newton (1972) argues that there was a sudden change in the properties of tides around the 7th or 8th century and suggests that an apparent change of the Normandy coastline early in the 8th century may be evidence for such a change. In view of the evidence that localized tidal friction may not be very important, such speculations appear inappropriate. More recent changes in tidal phase and amplitude, over the last $2\frac{1}{2}$ centuries, have been found by Cartwright (1971, 1972); about $5^\circ\ \text{cy}^{-1}$ in phase at the South Atlantic island of Saint Helena and $3^\circ\ \text{cy}^{-1}$ in Brest for the diurnal tides and about $1\ \%\ \text{cy}^{-1}$ in the amplitude of the M_2 tide at Brest. Whether these trends are truly oceanic or symptomatic of local tidal changes cannot be established with these data. The fact that (i) sea level has not varied greatly over the last few thousand years (Fairbridge 1961; Mörner 1971), that (ii) there has not been any significant change in the sea floor topography or in the ocean–continent distribution and that (iii) dissipation is apparently not controlled by pnenomena in a few localized regions, rules out any significant change in the lunar tidal acceleration over this time interval.

Changes in the secular rate of the Earth's spin can be readily accepted due to long period variations associated with angular momentum and inertia changes of the Earth and with torques acting on the mantle (Munk & MacDonald 1960). Muller & Stephenson (1975) and Muller (1975) discuss the astronomical evidence for such changes that may have occurred over the last 3000 years.

Integration back into time

It is invalid to integrate tidal equation (27) back into time because the interaction between the Moon and Sun is ignored. This is particularly important for the changes in inclination (see for example, Goldreich 1966). Nevertheless their integration does permit some conclusions to be drawn about the possible consequences of tidal dissipation in the oceans. Integrating these equations back into time with constant equivalent phase lag of 5° leads to the Moon's approach to within 10 Earth radii from the Earth about 1.5×10^9 years ago. With 5° for semi-diurnal, 2.5° for the diurnal and 0° for the long period tides this approach would have occurred some 1.7×10^9 years ago. The effect of diminishing the lag of the diurnal tides on the inclination of the orbital plane on the equator is not very important since the contributions from O_1 and K_1 tend to cancel each other.

Apart from changes in the ocean configuration, the equivalent phase lags of the tide K_1 (and of K_2) will vary with time due to their combined lunar and solar origin. The lunar part of $D^+_{22}\sin\epsilon^+_{22}$ will vary with time, but the solar part remains constant and the total K_1 tide will diminish in amplitude relative to, say, O_1. As these two tides contribute to di/dt with opposite signs, the variation of the inclination with time may be modified but the sign is unlikely to change since both the K_1 and O_1 contributions are smaller than the M_2 contribution. Possibly by postulating certain ocean basin resonances $(di/dt)_{O_1}$ may be made to dominate over $(di/dt)_{M_2}$, leading to a change of sign of di/dt.

As most of the dissipation occurs in the ocean it is improbable that the $D^+_{22}\sin\epsilon^+_{22}$ or $D^+_{21}\cos\epsilon^+_{21}$ have remained constant during the Earth's history. The equilibrium tide amplitude D^+_{22} is close to the observed value and is determined to within 10% solely by the a_{00} term in the ocean–continent expansion. As the total area of the continents has remained relatively constant throughout the geological past, D^+_{22} will not have varied by a significant amount. The phase, however, may have varied considerably and reduction by a factor of two in the equivalent lags will push the time of close approach to 3×10^9 years.

Without a clearer understanding of what controls the second degree phase lag of the ocean tides, it is inopportune to speculate on the possible consequences of continental unrest on the evolution of the Earth–Moon system. Yet, until this is understood and possible limits on past ocean tide parameters can be established, tidal dissipation theory will not impose any more stringent constraints on the Moon's origin than in does at present.

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